Demonstration of Rule XV p 54

Suppose $y=\phi(uv)$, where $u=\psi x$, $v=\chi x$

Put $uv=V$, then $y=\phi(V)$.

$$dy=\phi'(V)dV,$$
$$dV=u\,dv+v\,du,$$
$$du=\psi' x\,dx,$$
$$dv=\chi' x\,dx.$$

$$\therefore dy=\phi'(V)dV=\phi'(V)(u\chi' x\,dx+v\psi' x\,dx)$$

$$\frac{dy}{dx}=\phi'(V)(u\chi' x+v\psi' x)$$

But $\phi'(V)=\frac{dy}{dV}$, $\chi' x=\frac{dv}{dx}$, $\psi' x=\frac{du}{dx}$.

$$\therefore \frac{dy}{dx}=\frac{dy}{dV}\left(u\frac{dv}{dx}+v\frac{du}{dx}\right),$$
$$=u\frac{dy}{dV}\frac{dv}{dx}+v\frac{dy}{dV}\frac{du}{dx}\ \ldots(1)$$

Now if we suppose in $dV=d\,uv$, v to vary whilst u remains constant, and then u to vary whilst v remains constant, we have in the first case, for the value of dV, $u\,dv$, and in the second, $v\,du$. Hence, by substitution, (1) becomes

$$\frac{dy}{dx}=\frac{dy}{dv}\frac{dv}{dx}+\frac{dy}{du}\frac{du}{dx}.$$

AN ELEMENTARY TREATISE

ON THE

DIFFERENTIAL CALCULUS,

IN WHICH THE METHOD OF LIMITS IS EXCLUSIVELY MADE USE OF.

BY THE

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PREFACE.

THE method of Limits is generally allowed to be the best and most natural basis upon which to found the principles of the Differential Calculus; in the following pages this method is *exclusively* adopted, no use whatever being made of series in the demonstration of fundamental propositions. The following is an outline of the work, which is by no means offered to the reader as a complete treatise on the subject, but merely as an exposition of its more prominent and useful principles.

In Chap. I, certain terms, afterwards to be used, are defined and explained. In Chap. II the nature of a *Limiting Value* is fully set forth, and the important distinction (which ought never to be overlooked) between an *actual* and a *limiting* value is pointed out and illustrated by examples. Chap. III contains a set of *Lemmas*, which are necessary in order to render the use made of limiting values in the Differential Calculus perfectly legitimate; and here I have endeavoured to confine myself to what seems really essential. In Chap. IV certain important limiting values are obtained. Chap. V contains the *Rules for Differentiation*, in the demonstration of

which Lagrange's *functional* notation is employed, as being the simplest to begin with. In Chap. VI the *Differential notation* of Leibnitz is explained, $\frac{dy}{dx}$ is defined as the *quote* of the *differentials* dy and dx, which however are not supposed to be infinitesimals, but simply two arbitrary quantities in a certain ratio. In the case of *partial* differential coefficients, some modification of the common differential notation $\frac{du}{dx}$, $\frac{du}{dy}$, is clearly necessary: I have employed the suffix notation $d_x u$, $d_y u$, as being frequently employed, though not exactly in this manner. I should have much preferred the notation $d_x u$, $d_y u$ to denote partial *differentials*, and $\frac{d_x u}{dx}$ $\frac{d_y u}{dy}$ to denote partial *differential coefficients*. Chap. VII relates to *successive* differentiation, and the change of the *independant variable*. Chap. VIII contains certain very important *Lemmas* upon which the use and application of the Differential Calculus in a great measure depends. Chap. IX contains the theory of *Series*, based upon one of the preceding Lemmas, without assuming that $f(x+h)$ can be developed in the form

$$A + Bh^{\alpha} + Ch^{\beta} + \&c. \ldots :$$

and here I have endeavoured to shew what the real nature of a series is, and to prove rigorously the prin-

ciple of *Indeterminate Coefficients.* Chapters X and XI relate to *Vanishing Fractions, and Maxima and Minima,* and contain some useful simplifications of the common methods. The very insufficient and troublesome criterion usually employed in distinguishing the maxima and minima of functions of two variables is not introduced. Chap. XII relates to *Tangents, Normals,* &c., the *Curvature* of Curves, and the properties of the *Evolute;* and here the arrangement usually adopted is somewhat departed from, and what seems a more natural course pursued, in order to avoid certain difficulties, which I have observed very often impede the student on his first reading of the subject. In Chap. XIII the useful *Polar formulæ* and the differentials of *Areas, Volumes,* &c., are deduced. Chap. XIV relates to *Asymptotes.* Chap. XV contains a very simple method of *tracing curves.* Chap. XVI relates to *singular points.* Chap. XVII contains the general *Theory of Contacts* and *Ultimate Intersections:* no use is made of series in explaining the different orders of contact. The remaining chapters are occupied with *Elimination* by differentiation, *Lagrange's Theorem,* the properties of the *Cycloid,* &c., &c. The Appendix contains Examples worked out.

It was my intention to have added a few more chapters, and among the rest, one on the origin and progress of the Differential Calculus, and another on

the Infinitesimal method; but from various circumstances I found it impossible to send the work to the press at the time originally promised to my bookseller, without omitting these concluding chapters. I mention this to account for the absence of allusions to the History of the Differential Calculus, which were all reserved for the final chapter, and the small number of Examples in the Appendix.

Professor Peacock's excellent collection of Examples, which have been of such service to the Mathematical Student, is now out of print; but Mr Gregory's work lately published will supply its place, which contains, not only a great number of well-selected and valuable examples, but also many important explanations and theorems not to be met with in any elementary treatise. In a subject of so much importance as the present, the student ought not to confine his attention to one book or system: for a very valuable treatise on this subject he is referred to that published by the Society for the Diffusion of Useful Knowledge.

In the general plan of this work, and in several particulars, I have deviated from some of the methods often made use of, partly in attempting to put the subject in a simpler and clearer point of view, and partly in avoiding certain steps of reasoning which appear to be defective. One of these is the fallacy of establishing premises on a certain implied condition, and drawing a conclusion from them by a direct violation

of that condition. An example of this is to be found in a proof often given of the principle of indeterminate coefficients, in which the factor x is divided out of the equation

$$Bx + Cx^2 + Dx^3 + \&c. \ldots = 0,$$

which of course tacitly assumes the condition that x is not zero; in this manner is obtained the equation

$$B + Cx + Dx^2 + \&c. \ldots = 0;$$

and then by putting $x = 0$, contrary to the implied condition, the conclusion $B = 0$ is arrived at. Another example of this kind of reasoning is given in the first note, page 6.

The *assumption*, that $f(x+h)$ can be expanded in a series of the form $A + Bh^{\alpha} + Ch^{\beta}$ &c.... seems to me to be a serious defect in the common method of establishing Taylor's Series, and thereupon the principles of the Differential Calculus. This assumption is usually justified by arguing, that if we find definite values for A, B, C, &c. it shews that the assumption is correct. Now this argument may be stated thus:

"If the assumption that $f(x+h) = A + Bh^{\alpha} + Ch^{\beta}$ + &c. be true, then A, B, C, &c., must have definite values. But we can in general obtain definite values for A, B, C, &c.... (*e.g.* by the method of indeterminate coefficients.) Therefore the assumption is true."

This is clearly a falacious argument, for to warrant the conclusion the first premise should have been this:

"If the assumption *be not true*, definite values cannot be obtained for A, B, C, &c."

These defective steps of reasoning and others which might be mentioned, are objectionable, not because they lead to erroneous conclusions, but because they ought not to be found in a subject like the present, in which every thing should be comformable to the strictest rules of logical deduction. M. Cauchy has done much towards the improvement and perfection of the Differential Calculus, and his writings on this, like those on the more abstruse branches of mathematics, are most valuable. In one or two places the methods I have employed in the following pages are apparently similar to those of M. Cauchy, but in reality they are essentially different: so far as I am aware I am indebted to him only for article 48.

CAMBRIDGE,
October, 1842.

ERRATA.

The reader is requested to make the following correction which is of some importance.

3rd line from foot of page 75 instead of (2) *read* (0).

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THE

DIFFERENTIAL CALCULUS.

CHAPTER I.

PRELIMINARY REMARKS. VARIABLES AND CONSTANTS. FUNCTIONS, CONTINUOUS AND DISCONTINUOUS. ILLUSORY FUNCTIONS. EXAMPLES.

1. At the commencement of a treatise upon any science, the reader naturally expects to find some brief statement of the nature and object of that science; but in the present instance it is impossible to make such a statement, unless the meaning of certain terms, necessary to be used, be previously explained: for the Differential Calculus, from the very first, involves notions which must be new to one who is acquainted with no more than the common elements of Algebra and Analytic Geometry; and in the following pages we must suppose the reader to have advanced only so far in the study of mathematics. **The nature of the Differential Calculus cannot be explained at first.**

We shall therefore make a few preliminary remarks, and explain certain fundamental notions, before we state what the Differential Calculus is.

2. *Any quantity capable of variation is called a Variable.* **Variables and Constants.**

Any quantity which cannot vary, or, if it can, is supposed not to vary, is called a Constant.

Thus in the general equation to a right line, namely, $y = mx + c$, if we suppose the line never to change its position,

m and c are constants, and x and y are variables; but if we suppose that the line may change its position, m and c are variables as well as x and y, for they are then capable of variation.

It is usual to take the first letters of the alphabet a, b, c...&c. to denote constants, and the last letters...u, v, w, x, y, z to denote variables; but in many cases this distinction cannot be conveniently maintained, inasmuch as quantities which in one case we consider variable must often, in another, be considered constant, and vice versa.

Variables are often called, *Arbitrary quantities*, *Indeterminate quantities*, *Unassigned quantities*.

Continuous and Discontinuous Variables.

3. *A variable to which we may give in succession any number of different values, which differ from each other as little as we please, is called a Continuous Variable; otherwise it is said to be Discontinuous.*

Thus if x be taken to represent the length of any line which may be of any magnitude we please, then x is a continuous variable: but if x be taken to represent any *integer*, then it is a discontinuous variable; for in the former case we may give to x in succession any number of different values which differ from each other as little as we please; whereas in the latter case, though we may give x any number of different values, we cannot make them differ from each other as little as we please. The distinction between a continuous and a discontinuous variable may be briefly expressed by saying, that one admits of *gradual* change, but the other does not.

A Function defined. The Functional Notation explained.

4. *The result obtained by performing a certain operation or set of operations upon a variable* x, *is said to be a certain function of* x.

The words "*a certain function of*" are, for the sake of brevity, denoted by the letter f written before x. Thus the equation $y=f(x)$ simply means, that y is "*a certain function of*" x, i. e. that y is the result of performing a certain operation or set of operations upon x. The letter f, which we shall call the *Functional Letter*, is, as it were, the *represen-*

tative of the particular operation or set of operations performed upon x. Thus if $f(x) = x^n$, f represents the operation of taking a quantity to the n^{th} power; if $f(x) = \log \sin x$, f represents the operations of taking the logarithm of the sine of a quantity; if BPQ (fig. 1) be a given curve, $AM\ (= x)$ $MP\ (= y)$ the co-ordinates of any point P, and if we take $f(x)$ to represent y, then f represents the operations of measuring AM equal to x along the line AX, erecting MP perpendicular to AX, producing it to meet the curve at P, and so finding the length MP or y.

5. From the definition here given of a function we may see that $f(x)$ is not necessarily a quantity which changes when x changes; for a set of operations performed upon x may sometimes lead to a result which is the same for all values of x. This will appear from the following example. A function is not necessarily a quantity which changes when the variable changes.

Let APB (fig. 2) be a semicircle whose radius is a, C its center; take any arc $AP = x$, with P as center, and some given line c as radius, describe a circle cutting AB at the point Q: then we may say that AQ is a function of x and denote it by $f(x)$; f will therefore represent the operations of measuring x along AP, describing a circle with radius c and center P, so finding Q, and therefore finally AQ.

Now in general AQ or $f(x)$ will be different for different values of x; but if we take $c = a$, then the circle described with P as center and $c\ (= a)$ as radius will always cut AB at the center C, and therefore we shall have $AQ = AC$ or $f(x) = a$, whatever be the value of x. Hence $f(x)$ in this case does not vary when x varies.

Thus it appears, according to our definition of a function, that $f(x)$ is not necessarily a quantity which changes when x changes; and this remark is important, as will appear hereafter.

6. When we have occasion to consider several different functions at the same time, we employ *different* functional letters, in order to distinguish between them. The letters commonly used, in addition to f, are F, ϕ, ψ, χ, and sometimes these letters with dashes, thus f', F', ϕ', ψ', χ' or f'', F'', &c...&c. Thus we might put $f(x)$ to represent x^n, $\phi(x)$

to represent $\log \sin x$, $\psi'(x)$ to represent y in the curve, fig. 1, &c....&c.

Functions of several variables, as well as of one.

7. The result of performing a certain operation or set of operations upon *several* variables x, y, z, &c. is, in like manner, said to be a certain function of x, y, z, &c., and is represented similarly by the notation $f(x, y, z, \ldots)$. Thus $x^2 + y^2 + z^2$, $x \log (y + z)$, are the results of performing certain operations upon x, y, z, and are accordingly said to be certain functions of x, y, z, and we denote them by functional letters written before x, y, z, thus, viz.

$$x^2 + y^2 + z^2 = f(x, y, z) \qquad x \log (y + z) = \phi (x, y, z.)$$

Geometrical representation of a function.

8. We may evidently draw the curve BPQ (fig. 1) in such a manner that y shall be any function we please of x; and thus by means of a curve we may denote any function, and as it were represent it to the eye, which is often a very good method of illustrating general theorems respecting functions.

Functions of dependant and of independant variables.

9. In a function of several variables $f(x, y, z\ldots)$ it may happen that the variables x, y, $z\ldots$ are connected with each other in some manner, so that we cannot change one without at the same time changing the others. Or it may happen that x, y, $z\ldots$are not at all connected with each other, so that we may assign to each of them any value we please independently of the rest. In the former case $f(x, y, z\ldots)$ is said to be a function of several *mutually dependant* variables, and in the latter case $f(x, y, z)$ is said to be a function of several *independant* variables.

Functions explicit and implicit.

10. A quantity y is said to be an *explicit* function of another, x, when we can state the precise operations by which y may be deduced from x; if not, y is said to be an implicit function of x. Thus if we are given the equation

$$y^5 - 3x^4 y + x^5 = 0,$$

we know that there must be a certain set of operations by which y may be deduced from x, but what these operations are we cannot precisely state; in such a case y is called an implicit function of x.

The functions which commonly occur in mathematics are continuous.

11. The functions which commonly occur in mathematical investigations are of such a nature, that they always suffer a gradual and not a sudden change, so to speak, when the variable is gradually altered in value. Functions of this kind are called *Continuous Functions*. By saying that a function $f(x)$ always receives a gradual change when x is gradually varied, we mean this; that if x be changed into x', and therefore $f(x)$ into $f(x')$, then $f(x') \sim f(x)$ may be made as small as we please by taking $x' \sim x$* small enough, whatever be the value of x. That this is true for all ordinary functions, such as x^n, a^x, $\log x$, $\sin x$, &c....&c. and all ordinary combinations of these functions, such as $(a^x + \log \sin x)^n$, $a^x \sin x$, &c...&c. does not require to be proved, for it is quite evident. *All ordinary functions therefore are continuous functions.*

Discontinuous functions.

12. Functions which sometimes suffer a sudden change when the variable is gradually altered in value, are called *Discontinuous Functions*. Thus, if we draw a broken curve as in (fig. 3) the ordinate will be a discontinuous function of the abscissa: for it is evident that the ordinate will suffer a sudden change at the points P, Q, R, S, supposing the abscissa to be gradually varied.

13. We shall never have occasion to consider discontinuous functions in the following pages, and therefore we shall always suppose that the functions we make use of are continuous.

Assumption which we shall always make respecting functions.

Hence, if $f(x)$ be any function of x we make use of, and if x be changed into x', and therefore $f(x)$ into $f(x')$; we shall always assume that $f(x') \sim f(x)$ may be made as small as we please by taking $x' \sim x$ small enough whatever be the value of x.

An illusory function defined.

14. *A function* f(x) *is said to become Illusory when the operations represented by* f *cease to give any definite result;* which may happen in certain cases, as we shall shew.

* $x' \sim x$ denotes the difference between x' and x, subtracting the greater of these quantities from the lesser; $x' \sim x$ is therefore the absolute difference between x' and x without regard to sign.

Ex. 1. A fraction which assumes the form $\frac{0}{0}$.

Thus, if $f(x) = \dfrac{x^2 - x}{x^2 - 1}$, the operations represented by f cease to give any definite result when $x = 1$; for then $f(x)$ assumes the form $\dfrac{1-1}{1-1}$, or $\dfrac{0}{0}$, which is not a definite result, as is shewn in the note*; $f(x)$ therefore becomes illusory when $x = 1$.

Ex. 2. $(1+x)^{\frac{1}{x}}$ when $x=0$.

15. Again, if $f(x) = (1 + x)^{\frac{1}{x}}$, the operations represented by f cease to give a definite result when $x = 0$; for then $(1 + x)^{\frac{1}{x}}$ assumes the form $1^{\frac{1}{0}}$, which is not a definite result, as is shewn in the note†; $f(x)$ therefore becomes illusory, in this case, when $x = 0$.

Ex. 3. Case of two intersecting

16. Again, if ABC (fig. 4) be an ellipse marked with the usual letters, PG the line bisecting the angle SPH, and

$\frac{0}{0}$ does not represent any definite quantity.

* That $\frac{0}{0}$ is not a definite quantity appears thus. $\frac{0}{0}$ according to the strict definition of a quotient, is that quantity which multiplied by 0 gives 0: now any quantity whatever multiplied by 0 gives 0; therefore $\frac{0}{0}$ is any quantity whatever; i. e. it is not a definite quantity.

It may be said, however, that though $\frac{0}{0}$, considered absolutely, is not a definite quantity, nevertheless $\frac{x^2-x}{x^2-1}$ becomes $\frac{1}{2}$ when $x=1$; for $\frac{x^2-x}{x^2-1}=\frac{x}{x+1}$, and $\frac{x}{x+1}=\frac{1}{2}$ when $x=1$, and therefore $\frac{x^2-x}{x^2-1}=\frac{1}{2}$ when $x=1$.

To this we may answer, that $\frac{x^2-x}{x^2-1}$ is proved to be equal to $\frac{x}{x+1}$ by dividing its numerator and denominator by $x-1$; but we may not perform this division when $x-1=0$, since there is no rule of Algebra which enables us to divide the numerator and denominator of a fraction by zero without altering its value; hence the equation $\frac{x^2-x}{x^2-1}=\frac{x}{x+1}$ holds only on the express condition that x does not $=1$; and therefore we may not draw any conclusion from this equation which requires us to suppose that x actually $=1$: consequently, we cannot assert that $\frac{x^2-x}{x^2-1}=\frac{1}{2}$ when $x=1$ because $\frac{x^2-x}{x^2-1}=\frac{x}{x+1}$.

$1^{\frac{1}{0}}$ not a definite quantity.

† That $1^{\frac{1}{0}}$ is not a definite quantity may be shewn thus. $1^{\frac{1}{0}}$ is that quantity which taken to the power 0 becomes 1; now any quantity whatever taken to the power 0 becomes 1; therefore $1^{\frac{1}{0}}$ is not a definite quantity.

CM ($= x$) the abscissa of P; then if we assume $f(x) = CG$, the operations represented by f cease to give a definite result when $x = a$. For f represents the operations of drawing MP perpendicular to CM, so determining the point P, and the lines SP, HP, and then drawing PG to bisect $\angle SPH$, and so finding G, and therefore CG. Now when $x = a$, P coincides with A, and therefore the line PG with CA, and therefore PG cannot be said to intersect AC in one point more than another; therefore CG is not a definite quantity, and therefore the operations denoted by f cease to give a definite result when $x = a$. Hence $f(x)$ becomes illusory when $x = a$.

lines becoming coincident.

17. Again, if P (fig. 5) be a point on a curve PQ, RPQ a right line drawn through P and any other point Q on the curve, meeting the axis of x (AX) in the point R: then if we assume arc $PQ = s$, and $\angle PRX = \phi(s)$, the operations represented by ϕ cease to give a definite result when $s = 0$. For ϕ represents the operations of measuring $PQ = s$, drawing RPQ through P and Q, and so finding the angle PRX: now when $s = 0$, Q coincides with P, and any line whatever drawn through P passes also through Q; therefore the line QPR does not occupy a definite position, nor is $\angle PRX$ a definite angle, when Q coincides with P; and therefore the operations represented by f cease to give a definite result when $s = 0$. Hence $\phi(s)$ becomes illusory when $s = 0$.

Ex. 4. Case of a right line joining two points of a curve, when these points are made to coincide.

18. From these examples it is evident that a function may become illusory when x receives a particular value, the operations represented by the functional letter ceasing to give any definite result.

In each of these examples we may easily see that if x differ ever so little from that value which makes $f(x)$ illusory, the operations represented by f do lead to a definite result. It is therefore only for isolated values of x that $f(x)$ becomes illusory: and this will be found to be true in all cases where functions become illusory.

Functions become illusory only for isolated values of the variable.

CHAPTER II.

THE DISTINCTION BETWEEN AN ACTUAL AND A LIMITING VALUE EXPLAINED. A TANGENT DEFINED. THE NATURE OF THE DIFFERENTIAL CALCULUS STATED.

The distinction between an actual value and a limiting value.

19. THE Examples brought forward in the preceding chapter, for the purpose of shewing that a function may become illusory when the variable receives a particular value, lead us to make a very important though simple distinction, namely, the distinction between an *Actual Value* and a *Limiting Value* of a function.

An Actual Value of a function f(x) *is the result obtained by giving* x *some particular value, and performing upon it the operations represented by* f.

A Limiting Value of a function f(x) *is that quantity from which we may make* f(x) *differ as little as we please, by making* x *approach nearer and nearer in magnitude to some particular value without actually becoming equal to it.*

That the limiting value does not cease to be a definite quantity when the actual value does shewn by examples.

20. An actual and a limiting value thus defined seem at first sight to be the same thing; and indeed, as long as the operations represented by f lead to a definite result, they are identical, as we shall prove presently; it is only when the function becomes illusory that the distinction between them is real, and it consists in this, that *the actual value ceases to be a definite quantity, whereas the limiting value does not.* This we shall shew in the case of the examples just alluded to.

Ex. 1. A fraction which assumes the form $\frac{0}{0}$.

21. Let us consider the first example, namely

$$f(x) = \frac{x^2 - x}{x^2 - 1}, \quad \text{(see 14)}.$$

We have seen that the operations here denoted by f give no

definite result when $x = 1$, or, in other words, that the actual value ceases to be a definite quantity when $x = 1$.

But not so the limiting value; for we may divide the numerator and denominator of $f(x)$ by $x - 1$, and so arrive at the equation $f(x) = \frac{x}{x+1}$, except when x actually $= 1$; now by making x approach nearer and nearer in magnitude to 1, without actually becoming equal to it, we may evidently make $\frac{x}{x+1}$ differ as little as we please from $\frac{1}{2}$, and therefore the same may be said of $f(x)$, since it ceases to be equivalent to $\frac{x}{x+1}$ only when x actually $= 1$: $\frac{1}{2}$ is therefore a quantity from which we may make $f(x)$ differ as little as we please, by making x approach nearer and nearer in magnitude to 1, without actually becoming equal to it.

And it is easy to see that there is no other quantity but $\frac{1}{2}$ from which we may make $\frac{x}{x+1}$, and therefore $f(x)$, differ as little as we please by making x approach nearer and nearer to 1. Hence it appears that $\frac{1}{2}$, and no other quantity but $\frac{1}{2}$, is the limiting value of $f(x)$ when x approaches 1. We see therefore in this case that when the actual value ceases to be a definite quantity the limiting value does not.

22. We are not yet sufficiently advanced to shew that the same is true in the case of the second example (15); so we shall pass on to the third example (16). Ex. 2. Two intersecting lines which become coincident.

We have seen that the operations here represented by f cease to give a definite result when $x = a$, i. e. the actual value of $f(x)$ ceases to be a definite quantity when $x = a$. But not so the limiting value; for since PG bisects $\angle HPS$ we have

$$\frac{HG}{SG} = \frac{HP}{SP}, \quad \text{or} \quad \frac{ae + CG}{ae - CG} = \frac{a + ex}{a - ex} \quad \text{(by Conic Sections)};$$

$$\text{and } \therefore CG \text{ or } f(x) = e^2 x\,{}^*, \text{ except } x \text{ actually } = a.$$

* This proof evidently fails when $x = a$, for then P coincides with A and the triangle HPS ceases to exist.

Now $e^2 x$ may be made to differ from $e^2 a$ as little as we please by making x approach nearer and nearer in magnitude to a; therefore, as in the first example, $e^2 a$, and no other quantity but $e^2 a$, is the limiting value of $f(x)$ when x approaches a. Hence in this case when the actual value ceases to be a definite quantity the limiting value does not.

Ex. 3. A line cutting a curve in two points which become coincident.

23. Lastly, let us consider the fourth example (see 17).

We have seen that the operations here represented by ϕ cease to give a definite result when $s = 0$; i. e. the actual value of $\phi(s)$ ceases to be a definite quantity when $s = 0$. But not so the limiting value; for let $AM (= x)$ $MP (= y)$ be the co-ordinates of the point P (fig. 6), $AN (= x')$ $NQ (= y')$ the co-ordinates of the point Q, draw OP parallel to MN, and let $y = f(x)$ be the equation to the curve: then we evidently have

$$\tan PRX = \frac{OQ}{OP} = \frac{y' - y}{x' - x} = \frac{f(x') - f(x)}{x' - x},$$

and this equation is true, no matter how near x' may be to x, provided x' be not actually equal to x, for then the triangle OPQ ceases to exist, and $\tan PRX$ and $\dfrac{f(x') - f(x)}{x' - x}$ cease to have definite values.

Now for simplicity let us suppose the curve to be a parabola having the axis of y for its axis, and its equation being accordingly $y = \dfrac{x^2}{4m}$; then

$$\frac{f(x') - f(x)}{x' - x} = \frac{1}{4m}\,\frac{x'^2 - x^2}{x' - x} = \frac{x' + x}{4m},$$

except when $x' = x$.

Hence it is evident that, except when x' actually $= x$, we have

$$\phi(s) = \angle PRX = \tan^{-1}\frac{x' + x}{4m}.$$

Now the second member of this equation may be made to

differ from $\tan^{-1}\dfrac{x}{2m}$ as little as we please by making x' approach x, or, what is the same thing, by making s approach zero; hence (as in the first example) $\tan^{-1}\dfrac{x}{2m}$, and no other quantity, is the limiting value of $f(s)$ when s approaches zero.

In a similar manner we might shew, in the case of other curves, that $f(s)$ has a definite limiting value when s approaches zero.

It appears therefore in this case, that when the actual value ceases to be a definite quantity the limiting value does not.

24. This last example leads us to the best and most accurate conception of what a tangent is. For draw the line SPT making the angle $\tan^{-1}\dfrac{x}{2m}$ with the axis of x; then by what has been proved $\angle PTX$ is the limiting value of $\angle PRX$ when Q approaches P; i.e. $\angle PRX$ may be made to differ from $\angle PTX$ as little as we please by making Q approach P without actually coming up to it; or what is the same thing, $\angle QPS$ may be made as small as we please by making Q approach P without actually coming up to it. What a tangent to a curve is.

Now this being the case, it is natural to say that the line SPT just ***touches*** the curve at the point P, or that it is the ***tangent*** to the curve at P. Hence we define a tangent in the following manner.

25. If SPT be that line to which the secant RPQ may be made to approach nearer and nearer so as to make with it an angle as small as we please, by making Q approach P without actually coming up to it; then SPT is said to be the line touching the curve at the point P, or the tangent at P. Or to speak more briefly; *If* SPT *be the limiting position of the secant* RPQ *when* Q *approaches* P, *it is said to be the tangent at* P*. Definition of a tangent.

* This definition of a tangent seems to me to be the only accurate one that can be given, so as to apply to all cases of contact, such as contact at a point of con-

What the Differential Calculus is briefly stated.

26. The example we have just been considering, will enable us now to state generally the nature and object of the Differential Calculus.

We have seen, that to determine the position of the tangent at any point of a curve, as above defined, we have only to find the limiting value of $\frac{f(x') - f(x)}{x' - x}$ when x' approaches x; it appears therefore that we may arrive at a general method of drawing tangents to curves by means of this limiting value, if we can find it in all cases.

But this is a very small part of the use which may be made of this important limiting value: for there are very few branches of exact science which are not largely indebted to its assistance for the progress they have made. Indeed, without it, some of the most interesting applications of mathematics to the explanation of natural phænomena could never have been effected.

Now the Differential Calculus is that branch of mathematics whose object is, in the first place, to determine a set of rules whereby the limiting value of $\frac{f(x') - f(x)}{x' - x}$ when x' approaches x may be found with facility in all cases; and, in the second place, to explain some of the principal uses which may be made of this limiting value in pure mathematics.

The origin of the name *Differential Calculus* we shall presently explain.

trary flexure, contact at a cusp. It certainly is not correct to define the tangent STP to be the position which the line QPR assumes when Q coincides with P, for we have seen that the line QPR has no definite position when Q coincides with P.

CHAPTER III.

CERTAIN LEMMAS RESPECTING LIMITING VALUES.

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27. In the preceding Chapter it was our object to shew that there is a real distinction between an actual and a limiting value in certain cases, and to state briefly the nature of the Differential Calculus. We now proceed to prove certain Lemmas, and to obtain certain limiting values which we shall find useful hereafter. But we must make a few remarks previously.

The phrase "diminish ad libitum," explained.

28. When a quantity may be continually diminished, so as to become less than any specified quantity, however small, without becoming actually zero, we shall say that it may be "*diminished ad libitum.*" We shall find this phrase convenient and perhaps less likely to be misunderstood than the words "*diminish indefinitely,*" which are generally used in the same sense.

29. Employing this phrase, we may state the assumption made in (13) as follows: viz.

Assumption made in Art. 13, stated somewhat differently.

By continually diminishing $x' \sim x$, *when it has once become sufficiently small, we may diminish* $f(x') \sim f(x)$ ad libitum; $f(x)$ *being any function, and* x *any value of the variable. And more generally by continually diminishing* $x' \sim x$, $y' \sim y$, $z' \sim z$ ... *when they have once become sufficiently small, we may diminish* $f(x'y'z'...) \sim f(xyz...)$ ad libitum; $f(xyz...)$ *denoting any function of several variables, and* x, y, z... *any values of these variables.*

Of course we here suppose that $f(x)$ $\{$or $f(x, y, z...)$ if there be more than one variable$\}$ is not illusory.

30. And the definition of a limiting value in (19) may be stated thus:

Definition of a limiting value stated somewhat differently.

If by continually diminishing x ~ a, *when it has once become sufficiently small, we may diminish* f(x) ~ A ad libitum; *then* A *is said to be the limiting value of* f(x), *when* x *approaches* a.

It is clear that the assumption and definition thus stated are equivalent to what they were before, only that they are more exactly expressed and better adapted to the use we shall have to make of them hereafter.

$f(x) \sim A$ does not in general diminish with $x \sim a$ until $x \sim a$ has become less than a certain value.

31. It is important to notice, that in this definition of a limiting value we do not assert that $f(x) \sim A$ must diminish when $x \sim a$ is diminished, for all values of $x \sim a$, but only when $x \sim a$ has once become sufficiently small. There are many cases in which $f(x) \sim A$ increases as $x \sim a$ diminishes, and continues to increase until $x \sim a$ has become less than a certain value, after which it continually diminishes with $x \sim a$.

To avoid circumlocution, instead of saying, "By diminishing $x \sim a$, when once it has become sufficiently small, we diminish $f(x) \sim A$ *ad libitum*," we shall simply say, "By *sufficiently* diminishing $x \sim a$ we diminish $f(x) \sim A$ *ad libitum*."

x may be a discontinuous variable, provided it be possible to diminish $x \sim a$ ad libitum.

32. It is also important to observe, that it is not essential to our conception of a limiting value, as above defined, that x should be a continuous variable (see 3); all that is necessary is this, that it be possible to diminish $x \sim a$ *ad libitum*, that is, to make it less than any specified quantity however small, without actually making it zero.

Thus if we suppose $x = a + \frac{1}{n}$, where n is always an integer, and therefore x not a continuous variable, we may conceive the existence of a limiting value of $f(x)$ when x approaches a (which it does when n approaches infinity), just as well as if x varied continuously: for by increasing n, though it be an integer, we may diminish $\frac{1}{n}$ or $x \sim a$, *ad libitum;*

which is all that is supposed necessary so far as $x \sim a$ is concerned in our definition of a limiting value.

But if we suppose $x = a + n$, where n is always an integer, then it is impossible to conceive the existence of a limiting value of $f(x)$ when x approaches a, since we cannot make $x \sim a$ or n less than unity unless we make it actually zero.

We now proceed, as we stated, to prove certain Lemmas, which will enable us to reason more exactly hereafter, and will now serve to illustrate the nature of limiting values.

33. *The limiting value of* f(x) *when* x *approaches* a *is* Lemma I.
the same whatever sort of variable x *be, provided of course* $x \sim a$ *may be diminished* ad libitum.

Let x and $\mathfrak{x}$ be two different variables such that both $x \sim a$ and $\mathfrak{x} \sim a$ may be diminished *ad libitum*, and let A be the limiting value of $f(x)$ when x approaches a. Then by *sufficiently* diminishing $x \sim a$ and $\mathfrak{x} \sim a$, and therefore $\mathfrak{x} \sim x$, we diminish *ad libitum*, $f(x) \sim A$ by the definition in (30), $f(x) \sim f(\mathfrak{x})$ by the assumption in (29), and therefore $f(\mathfrak{x}) \sim A$. Hence by *sufficiently* diminishing $\mathfrak{x} \sim a$ we diminish $f(\mathfrak{x}) \sim A$ *ad libitum*, and therefore, by the definition (30), A is the limiting value of $f(\mathfrak{x})$ when $\mathfrak{x}$ approaches a, as well as that of $f(x)$ when x approaches a: from which the truth of the Lemma is manifest.

Thus if x be a continuous variable, and $\mathfrak{x}$ a discontinuous Examples.
variable in the form $a + \frac{1}{n}$, the limiting value of $f(x)$ when x approaches a, is also that of $f(\mathfrak{x})$ when $\mathfrak{x}$ approaches a.

Again, the limiting value of $f(x)$ when x approaches a is just the same thing as that of $f(e^x)$ or of $f(\tan x)$ when e^x or $\tan x$ respectively approaches a.

34. *If* f(x), *when expressed in terms of another variable* Lemma II.
z, *becomes* ϕ(z), *and if* x = a *when* z = b; *then the limiting value of* f(x) *when* x *approaches* a, *and the limiting value of* ϕ(z) *when* z *approaches* b, *are the same thing.*

For let A be the former limiting value; then, since $x = a$

when $z = b$, it is evident that by *sufficiently* diminishing $z \sim b$ we may diminish $x \sim a$ *ad libitum*, and therefore $f(x) \sim A$, by definition (30), and therefore $\phi(z) - A$ since $\phi(z) = f(x)$; therefore A is the limiting value of $\phi(z)$ when z approaches b, by def. (30). Q. E. D.

Examples. Thus if in the function $f(x)$ we put $x = a + \frac{1}{z}$, and therefore $f(x) = f\left(a + \frac{1}{z}\right) = \phi(z)$ suppose; then $x = a$ when $z = \infty$, and therefore the limiting value of $f(x)$ when x approaches a is the same thing as that of $\phi(z)$ or $f\left(a + \frac{1}{z}\right)$ when z approaches ∞.

Again, if $f(x) = \frac{\epsilon^x - 1}{x}$, and we put $\epsilon^x = z$, and therefore $x = \log z$, and therefore $f(x) = \frac{z-1}{\log z} = \phi(z)$ suppose; then x evidently $= 0$ when $z = 1$, and therefore the limiting value of $f(x)$ or $\frac{\epsilon^x - 1}{x}$ when x approaches 0, and that of $\phi(z)$ or $\frac{z-1}{\log x}$ when z approaches 1, are the same thing.

We shall find this method of transforming expressions very useful in finding their limiting values.

Lemma III. 35. *Every Actual value is also a Limiting value.*

We have defined $f(a)$ to be the result of performing the operations represented by f upon a; therefore, by the definition of an actual value in (19), $f(a)$ represents the actual of $f(x)$ when $x = a$. But, from the assumption stated in (29), and the definition of a limit in (30), it is evident that $f(a)$ is the limiting value of $f(x)$ when x approaches a. Hence the truth of the lemma is evident. Of course we here suppose that $f(a)$ is not illusory.

Lemma IV. 36. *If* A *be the Limiting value of* f (x) *when* x *ap-*

proaches a, f (x) *has the same sign as* A *for all values of* x *taken sufficiently near* a.

Since $f(x) \sim A$ may be diminished *ad libitum* by taking x near enough to a, it is clear that $f(x)$ may be made greater or less than zero, according as A is greater or less than zero; or in other words, $f(x)$ may be made to have the same sign as A by taking x near enough to a: and when x is made to approach still nearer to a, since we so diminish $f(x) \sim A$ still more, at least for all values of x near enough to a (see 31), $f(x)$ must *continue* to have the same sign as A. Hence $f(x)$ has the same sign as A for all values of x taken near enough to a. Q. E. D.

37. *If* f (x) *be any function of* x *which becomes illusory when* x = *a certain value* a, *and if for each value of* x (a *of course excepted*) f (x) *has only one value; then* f (x) *cannot have more than one limiting value when* x *approaches* a. Lemma V.

If possible let two different quantities A and B be both limiting values of $f(x)$ when x approaches a; then we may make $f(x)$ differ from both A and B as little as we please, at the same time, by *sufficiently* diminishing $x \sim a$; therefore x may be so taken, that A and B shall differ from the same quantity $f(x)$ {since $f(x)$ has only one value}, and therefore from each other, as little as we please; which is absurd if A and B be two different quantities; therefore A must be equal to B. Hence there cannot be more than one limiting value. Q. E. D.

38. Hence if we can prove that A is a limiting value of $f(x)$ when x approaches a, we are sure that no other quantity but A is a limiting value, and therefore that A is *the* limiting value.

Hence it appears that a limiting value is not a mere approximation, but a perfectly definite quantity; for if it were a mere approximation, then, when we had found a limiting value A, any quantity differing very little from A would be just as much a limiting value as A; contrary to what has been just proved. A limiting value is not a mere approximate quantity.

We have supposed that the function $f(x)$ has only one value for each value of x; if however it has more than one, n values suppose, it is evident that there will be n different limiting values, and no more, when x approaches a. Thus if

$$f(x) = a + b\left(2\frac{x^2 - x}{x^2 - 1}\right)^{\frac{1}{2}}$$

which has two values for each value of x, then there are two limiting values when x approaches 1; viz. $a + b$ and $a - b$.

Lemma VI. 39. *If* f(x) *and* ϕ(x) *be two functions, one of which,* f(x), *becomes illusory when* x = *a certain value* a, *and the other,* ϕ(x), *does not; and if we can shew that* f(x) = ϕ(x) *for all values of* x, a *of course excepted; then* ϕ(a) *is the limiting value of* f(x) *when* x *approaches* a.

For by (29), we diminish $\phi(x) \sim \phi(a)$ *ad libitum* by *sufficiently* diminishing $x \sim a$; but in so doing we never suppose x to become actually equal to a, and we are therefore sure that $f(x) = \phi(x)$; therefore, we diminish $f(x) \sim \phi(a)$ *ad libitum*, by *sufficiently* diminishing $x \sim a$; i. e. $\phi(a)$ is the limiting value of $f(x)$ when x approaches a Q.E.D.

Cor. 1. If $\phi(a)$ be illusory as well as $f(a)$, and if we know A to be the limiting value of $\phi(x)$ when x approaches a; then we may shew, in exactly the same way, that A is also the limiting value of $f(x)$ when x approaches a.

Cor. 2. If, instead of being able to shew that $f(x) = \phi(x)$ for all values of x except a, we can prove that $f(x) \sim \phi(x)$ is diminished *ad libitum* by *sufficiently* diminishing $x \sim a$; then the same conclusions evidently follow; that is to say; the limiting value of $f(x)$ when x approaches a is $\phi(a)$, or A if $\phi(a)$ be illusory.

Lemma VII. 40. *If* f(x) *be a function which becomes illusory when* x = a, *and if we can prove that* f(x) *lies between* another function* ϕ(x) *and a constant* A *for all values of* x *taken sufficiently near* a; *and moreover, that* A *is the limiting*

* When we say that $f(x)$ lies between $\phi(x)$ and A, we mean that $f(x)$ is not greater than one of these quantities and not less than the other

value of $\phi(x)$ *when* x *approaches* a*; then* A *is also the limiting value of* f(x) *when* x *approaches* a.

For since A is the limiting value of $\phi(x)$ when x approaches a, $\phi(x) \sim A$ is diminished *ad libitum* when $x \sim a$ is *sufficiently* diminished; therefore, *à fortiori*, since $f(x)$ lies between $\phi(x)$ and A, $f(x) \sim A$ is also diminished *ad libitum* at the same time; therefore A is the limiting value of $f(x)$ when x approaches a. Q.E.D.

41. *If* U, V, W... *be any functions of* x, *and* f(U, V, W...) *any function of these functions, and if* A, B, C... *be the limiting values of* U, V, W... *respectively when* x *approaches a certain value* a*; then* f(A, B, C...) *is the limiting value of* f(U, V, W...) *when* x *approaches* a. Lemma VIII.

For by *sufficiently* diminishing $x \sim a$ we diminish $U \sim A$, $V - B$, $W - C$... *ad libitum*, and therefore, by the assumption in (29), we diminish $f(U, V, W...) \sim f(A, B, C...)$ *ad libitum;* therefore $f(A, B, C...)$ is the limiting value of $f(U, V, W...)$ when x approaches a.

Thus the limiting value, when x approaches a, Examples.

of $U \pm V$ is $A \pm B$,

that of UV is AB,

that of $\dfrac{U}{V}$ is $\dfrac{A}{B}$,

that of $(U^2 + V^2) \sin W$ is $(A^2 + B^2) \sin C$.

In the proof of this Lemma, since we only speak of the *limiting* values of W, V, W... we make no supposition as to whether U, V, W... become illusory or not when $x = a$; so that the Lemma is equally true whether they do or whether they do not. If any of these quantities, U suppose, does not become illusory when $x = a$, then A is of course its actual value.

This Lemma evidently fails when the substitution of one or more of the quantities A, B, C... for U, V, W... respectively makes f illusory.

Example of the use of Lemma VIII. combined with some of the preceding Lemmas.

42. The following example will shew the use of Lemma VIII. combined with some of the preceding Lemmas.

Let $U = \dfrac{x^2 - 3x + 2}{x^2 - 1}$, and $V = \dfrac{2^{2x} - 2^{x+1}}{2^{2x} + 2^x - 6}$,

and suppose we wish to find the limiting value of

$$\frac{U^2 + UV + 10V^2}{10V - U} \sin^{-1}\{(U + 7) V\},$$

when x approaches 1.

Then, dividing out $x - 1$, $U = \dfrac{x - 2}{x + 1}$ except when $x = 1$, and $\dfrac{x - 2}{x + 2} = -2$ when $x = 1$. Therefore by Lemma VI. -2 is the limiting value of U when x approaches 1.

Again putting $2^x = z$, V becomes $\dfrac{z^2 - 2z}{z^2 + z - 6}$, and $x = 1$ when $z = 2$; hence by Lemma II. the limiting value of V when x approaches 1, is the same thing as that of $\dfrac{z^2 - 2z}{z^2 + z - 6}$, when z approaches 2. Now except when $z = 2$,

$$\frac{z^2 - 2z}{z^2 + z - 6} = \frac{z}{z + 3} \text{ which } = \tfrac{1}{5} \text{ when } z = 2;$$

therefore by Lemma VI. $\frac{1}{5}$ is the limiting value of V when x approaches 1. Hence by Lemma VIII. the limiting value of

$$\frac{U^2 + UV + V^2}{10V - U} \sin^{-1}\{(U + 7) V\},$$

when x approaches 1 is

$$\frac{4 - \frac{2}{5} + \frac{10}{25}}{\frac{10}{5} + 2} \sin^{-1}\{(-2 + 7)\tfrac{1}{5}\} = \frac{5\pi}{2}.$$

43. In proving the above Lemmas we have supposed, that functions become illusory only for isolated values of the variable; or in other words, that, if $f(a)$ be illusory, $f(x)$

is not illusory for all the values of x extremely near a, but only for the single value a. This assumption is evidently true for all ordinary functions.

We have not assumed the existence of a limiting value when the function becomes illusory in any of these Lemmas, inasmuch as they are worded in this manner; "*If* A be the limiting value then such and such things follow": nor shall we have any occasion to make this assumption; for in all cases where we have to consider the limiting values of functions when they become illusory, the existence of such values will be proved and not assumed. We may just observe however, that the existence of limiting values of functions when they become illusory is a necessary consequence of their becoming illusory only for isolated values of the variable, and of the assumption in (29), as it is not difficult to see.

CHAPTER IV.

CERTAIN LIMITING VALUES DETERMINED WHICH WE SHALL REQUIRE TO KNOW HEREAFTER.

Lemma IX. 44. If s *be the length of any arc of a curve, and* c *the length of its chord; the limiting value of* $\frac{s}{c}$ *when* s *approaches zero is unity.*

Let PSQ be the arc (fig. 7), PCQ the chord, draw PR the tangent at P, and QR perpendicular to PR, and let $\angle RPQ = \omega$: then, as long as P does not actually coincide with Q, we may evidently assume that PSQ lies between PCQ and $PR + QR$ (supposing of course that Q is taken near enough to P, so that the curve shall always bend towards the same side between P and Q); i.e. we may assume that

$$s \text{ lies between } c \text{ and } c\cos\omega + c\sin\omega;$$

$$\text{and therefore that } \frac{s}{c} \text{ lies between } 1 \text{ and } \cos\omega + \sin\omega.$$

Now, by the definition of a tangent in (25), 0 is the limiting value of ω, and therefore 1 that of $\cos\omega + \sin\omega$, when s approaches 0: therefore by Lemma VII, 1 is the limiting value of $\frac{s}{c}$ when s approaches 0. Q.E.D.

Lemma X. 45. *If* θ *be an angle measured by the subtending arc of a circle, the limiting value of* $\frac{\theta}{\sin\theta}$, *and that of* $\frac{\theta}{\tan\theta}$, *when* θ *approaches zero, are each unity.*

We may evidently assume, for all values of θ less than $\frac{\pi}{2}$, that

$$\theta \text{ lies between } \sin\theta \text{ and } \tan\theta,$$

and therefore that $\dfrac{\theta}{\sin\theta}$ lies between 1 and $\dfrac{1}{\cos\theta}$.

Now $\dfrac{1}{\cos\theta} = 1$ when $\theta = 0$; hence by Lemma VII. 1 is the limiting value of $\dfrac{\theta}{\sin\theta}$ when θ approaches 0. Q. E. D.

46. We might have deduced this result from the previous Lemma, by putting $s = 2\theta$, and c (which now becomes the trigonometrical chord of s) $= 2\sin\theta$, and therefore

$$\frac{s}{c} = \frac{\theta}{\sin\theta}.$$

47. *To find the limiting value of* $\dfrac{x^n - 1}{x - 1}$ *when* x *approaches unity,* n *being any number positive or negative integral or fractional.* Lemma XI.

It is clear that, whether n be positive or negative fractional or integral, it may always be expressed in the form $\dfrac{p-q}{r}$, where p, q, and r are positive integers. Hence we may put

$$\frac{x^n - 1}{x - 1} = \frac{x^{\frac{p-q}{r}} - 1}{x - 1} = \frac{z^{p-q} - 1}{z^r - 1} \text{ if we put } x = z^r$$

$$= \frac{1}{z^q}\,\frac{(z^p - 1) - (z^q - 1)}{z^r - 1}.$$

Hence, dividing $(z^p - 1) - (z^q - 1)$ and $(z^r - 1)$ by $z - 1$, observing that p, q, r are positive integers, we have

$$\frac{x^n - 1}{x - 1} = \frac{1}{z^q} \cdot \frac{(1 + z + z^2 \ldots\ldots p \text{ terms}) + (1 + z + z^2 \ldots\ldots q \text{ terms})}{1 + z + z^2 \ldots\ldots r \text{ terms}}$$

except of course when $z = 1$.

Now when $x = 1$, $z = 1$, and therefore the actual value of the second member of this equation is evidently $\frac{p-q}{r}$ or n: hence, by Lemma VI, n is the limiting value of $\frac{x^n - 1}{x - 1}$ when x approaches 1.

Lemma XII.

48. *To shew that there exists a limiting value of* $\left(1 + \frac{1}{n}\right)^n$ *when* x *approaches infinity,* n *being always an integer.*

By the Binomial Theorem for positive integers we have

$$\left(1 + \frac{1}{n}\right)^n = 1 + \frac{n}{1}\frac{1}{n} + \frac{n.n-1}{1.2}\frac{1}{n^2}\text{......to } (n+1) \text{ terms}$$

$$= 1 + 1 + \frac{1}{\Gamma 2}\left(1 - \frac{1}{n}\right) + \frac{1}{\Gamma 3}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\text{.........(1)},$$

where $\Gamma 2$, $\Gamma 3$, ... and in general Γr, denote (1.2), (1.2.3), ... (1.2.3...r), respectively.

Now in the series (1) all the terms are evidently positive, and when n increases each term increases, and the number of terms also increases. Moreover, this series is term by term less than the series

$$1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} \text{.... to } n + 1 \text{ terms},$$

which is $< 1 + \frac{1}{1 - \frac{1}{2}}$* or 3.

Hence $\left(1 + \frac{1}{n}\right)^n$, which $= 2$ when $n = 1$, continually in-

* A geometric series $1 + z + z^2$......to p terms $= \frac{1 - z^p}{1 - z} = \frac{1}{1 - z} - \frac{z^p}{1 - z}$. Now if z be positive and < 1, $\frac{z^p}{1 - z}$ is a positive quantity, and therefore the series $= \frac{1}{1 - z} -$ a positive quantity; therefore it is $< \frac{1}{1 - z}$, and this is true no matter how large p may be. *Hence, if z be a positive quantity less than unity,* $1 + z + z^2 + z^3$......... *continued to any number of terms is* $< \frac{1}{1 - z}$.

creases as n increases, but never comes up to 3, no matter how large n becomes. Therefore it is clear that there must be some number between 2 and 3 to which we may make $\left(1+\frac{1}{n}\right)^n$ *approach *ad libitum* by making n approach ∞, which number is the limiting value whose existence we wish to prove.

49. *To determine numerically the limiting value of* $(1+n)^{\frac{1}{n}}$ *when* n *approaches* ∞. Lemma XIII.

The $(r+1)^{\text{th}}$ term of the series (1) in the last article is

$$\frac{1}{\Gamma r}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)(\ldots\ldots)\left(1-\frac{r-1}{n}\right),$$

which is clearly $<\frac{1}{\Gamma r}$.

Hence the $(r+1)^{\text{th}}$ and following terms of the series (1) form a positive quantity less than

$$\frac{1}{\Gamma r}+\frac{1}{\Gamma(r+1)}+\frac{1}{\Gamma(r+2)}\ldots\ldots$$

which is $<\frac{1}{\Gamma r}\left(1+\frac{1}{r}+\frac{1}{r^2}\ldots\ldots\right)$,

which (by note, 48) is

$$<\frac{1}{\Gamma r}\left(\frac{1}{1-\frac{1}{r}}\right),\quad\text{or}\quad\frac{1}{\Gamma(r-1)}\cdot\frac{1}{r-1},$$

which is $<\frac{1}{\Gamma(r-1)}$.

Hence $$\left(1+\frac{1}{n}\right)^n=1+1+\frac{1}{\Gamma 2}\left(1-\frac{1}{n}\right)$$
$$+\frac{1}{\Gamma 3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\text{ to } r \text{ terms}$$
$$+\text{ a positive quantity less than }\frac{1}{\Gamma(r-1)}.$$

* By saying that one quantity u *approaches* another v *ad libitum*, we simply mean that $u\sim v$ diminishes *ad libitum*.

If therefore we suppose $r = 12$, and consider only the first seven decimal places, it is clear from the table in the note*, that

$$\left(1 + \frac{1}{n}\right)^n = 1 + 1 + \frac{1}{\Gamma 2}\left(1 - \frac{1}{n}\right)$$

$$+ \frac{1}{\Gamma 3}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \ldots\ldots \text{ to } 12 \text{ terms};$$

and this is true, no matter how large n may be.

Now by making n approach ∞, the second term of this equation may, term by term, be made to approach *ad libitum* to the series

$$1 + 1 + \frac{1}{\Gamma 2} + \frac{1}{\Gamma 3} \ldots\ldots \text{ to } 12 \text{ terms};$$

which to 7 decimal places = 2.7182818 (by table, *).

*

$1 =$	1.00000000	
$\frac{1}{\Gamma 2} =$	.50000000	
$\frac{1}{\Gamma 3} =$	.16666666	
$\frac{1}{\Gamma 4} =$	.04166666	
$\frac{1}{\Gamma 5} =$	.00833333	
$\frac{1}{\Gamma 6} =$	.00138888	This table is easily formed by dividing 1.00000000 by 2, the result by 3, the result so obtained by 4, and so on
$\frac{1}{\Gamma 7} =$	.00019841	
$\frac{1}{\Gamma 8} =$	00002480	
$\frac{1}{\Gamma 9} =$	.00000275	
$\frac{1}{\Gamma 10} =$	.00000027	
$\frac{1}{\Gamma 11} =$	.00000002	

Sum to seven decimal places = .7182818.

Hence if we consider only the first seven decimal places, 2.7182818 is the limiting value of $\left(1+\frac{1}{n}\right)^n$, when n approaches ∞.

50. In the same manner we might obtain this limiting value to any greater number of decimal places. It is, however, an incommensurable number like π, and therefore, no matter how far we go, we shall never be able to obtain it exactly in numbers. It is usually denoted by the letter e, and it is taken as the base of a system of logarithms commonly called *Hyperbolic.* Base of Hyperbolic Logarithms, what.

For a certain reason, which we shall presently explain, logarithms calculated to base e are the most convenient to use in analytical calculations. We shall therefore always suppose that e is the base of whatever logarithms we have occasion to employ unless the contrary be specified.

51. *Hence* e *is the limiting value of* $(1+x)^{\frac{1}{x}}$ *when* x *approaches zero,* x *being any continuous variable.* Cor. 1.

For this limiting value by Lemma I. is the same thing as the limiting value of $(1+z)^{\frac{1}{z}}$ when z approaches 0, where $z=\frac{1}{n}$ and n is always an integer; and this latter limiting value by Lemma II. is the same thing as that of $\left(1+\frac{1}{n}\right)^n$ when n approaches ∞, which $= e$. Q. E. D.

52. *Hence,* $log_a e$ *or* $\frac{1}{log\, a}$* *is the limiting value of* $\frac{log_a(1+x)}{x}$ *when* x *approaches zero.* Cor. 2.

For $\frac{\log_a(1+x)}{x}=\log_a(1+x)^{\frac{1}{x}}$, and by what has just been proved, and by Lemma VIII. the limiting value of this latter quantity when x approaches zero is $\log_a e$.

* If $\log_a e = c$, then $a^c = e$, and $\therefore c \log_e a = 1$, or $c = \frac{1}{\log a}$.

Cor. 3. 53. *Hence also,* $\frac{1}{log_a e}$ *or log* a *is the limiting value of* $\frac{a^x - 1}{x}$ *when* x *approaches zero.*

For, put $a^x - 1 = z$, and $\therefore\ x = \log_a(1 + z)$;

$$\text{then } \frac{a^x - 1}{x} = \frac{z}{\log_a(1 + z)} = \frac{1}{\log_a(1 + z)^{\frac{1}{z}}}.$$

Now $x = 0$ when $z = 0$; hence by Lemma II, the limiting value of $\frac{a^x - 1}{x}$ when x approaches 0, is the same thing as that of $\frac{1}{\log_a(1 + z)^{\frac{1}{z}}}$ when z approaches 0 which, by what has been just proved, $= \frac{1}{\log_a e}$ or $\log a$.

CHAPTER V.

RULES FOR FINDING THE LIMITING VALUE OF $\dfrac{f(x') - f(x)}{x' - x}$.

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54. We now enter upon the Differential Calculus, properly so called, which, as we have stated, is that branch of mathematics whose object is, in the first place, to determine a set of Rules whereby the limiting value of $\dfrac{f(x') - f(x)}{x' - x}$ when x' approaches x may be found in all cases with facility; and in the second place, to shew and explain some of the principal uses which may be made of this limiting value in pure mathematics.

We proceed therefore, in the first place, to determine a set of Rules whereby this limiting value may be found in all cases with facility.

55. *We shall represent the limiting value of* $\dfrac{\mathrm{f}(\mathrm{x}') - \mathrm{f}(\mathrm{x})}{\mathrm{x}' - \mathrm{x}}$ *when* x' *approaches* x *by the notation* $\mathrm{f}'(\mathrm{x})$; *i. e. by simply dashing the functional letter.* Notation by which we represent the limiting value of $\dfrac{f(x')-f(x)}{x'-x}$

Thus if $f(x) = x^2$, $\dfrac{f(x') - f(x)}{x' - x} = \dfrac{x'^2 - x^2}{x' - x} = x' + x$, Examples.

except when x' actually $= x$; now the actual value of $x' + x$ when $x' = x$, is $2x$; hence, by Lemma VI., the limiting value of $\dfrac{x'^2 - x^2}{x' - x}$ when x' approaches x is $2x$; i.e.

$$\text{if } f(x) = x^2, \text{ then } f'(x) = 2x.$$

Again if $f(x) = ax + bx^3$, $\dfrac{f(x') - f(x)}{x' - x}$

$$= \frac{a(x' - x) + b(x'^3 - x^3)}{x' - x} = a + b(x'^2 + x'x + x^2);$$

hence, as in the previous case, we find that if

$$f(x) = ax + bx^3, \quad \text{then } f'(x) = a + 3bx^2.$$

And in general we may very easily shew in the same manner that if

$$f(x) = a_0 + a_1 x + a_2 x^2 \ldots a_n x^n,$$

$$\text{then } f'(x) = a_1 + 2a_2 x + 3a_3 x^2 \ldots + n a_n x^{n-1}.$$

Thus when $f(x)$ is a rational and integral function of x, $f'(x)$ may be *derived* from it by multiplying each power of x by its index, and lowering that index by unity.

Origin of the notation $f'(x)$, and of the name Derivative or Derived Function.

56. This notation f' is due to Lagrange: the function $f'(x)$ was called by him the *Derived Function* or *Derivative* of $f(x)$, because it may be derived from $f(x)$ by the peculiar process just stated, at least when $f(x)$ is a rational and integral function of x.

The following are the Rules whereby we may find this derived function or derivative in all cases with facility.

Rule 1. $f(x)=c$.

57. *If* f(x) = *a constant* c*, f′(x) = 0.

For then $\dfrac{f(x') - f(x)}{x' - x} = \dfrac{c - c}{x' - x} = 0$, except $x' = x$;

$\therefore$ lim. val. of $\dfrac{f(x') - f(x)}{x' - x}$ when x' app. x is zero, by Lem. VI.,

$$\text{i.e. } f'(x) = 0.$$

Cor. If $f(x) = c + \phi(x)$, $f'(x) = \phi'(x)$.

For then $\dfrac{f(x') - f(x)}{x' - x} = \dfrac{\phi(x') - \phi(x)}{x' - x}$,

and $\therefore f'(x) = \phi'(x)$. by Lem. VI. Cor. 1.

A constant therefore added to a function does not appear in the derived function.

* We have seen in (5) that a function of x may be a constant quantity.

58. *If* $f(x) = c\phi(x)$, ϕ *denoting any function; then* $f'(x) = c\phi'(x)$. Rule II. $f(x) = c\phi(x)$.

For then $\dfrac{f(x') - f(x)}{x' - x} = c\,\dfrac{\phi(x') - \phi(x)}{x' - x}$,

the limiting value of the second member of this equation when x' approaches x is $c\phi'(x)$: hence, by Lem. VI. Cor. 1,

$$f'(x) = c\phi'(x).$$

59. *If* $f(x) = \phi(x) \pm \psi(x) \pm \chi(x) \pm$ &c., ϕ, ψ, χ, &c., *denoting any functions; then* Rule III. $f(x) = \phi(x) \pm \psi(x)\ldots$

$$f'(x) = \phi'(x) \pm \psi'(x) \pm \chi'(x) \pm \text{\&c.}$$

For then $\dfrac{f(x') - f(x)}{x' - x} = \dfrac{\phi(x') - \phi(x)}{x' - x} \pm \dfrac{\psi(x') - \psi(x)}{x' - x}$

$$\pm \frac{\chi(x') - \chi(x)}{x' - x} \pm \ldots \text{\&c.}$$

The limiting value of the second member of this equation when x' approaches x is

$\phi'(x) \pm \psi'(x) \pm \chi'(x) \pm \ldots$ &c., by Lemma VIII.;

hence, $f'(x) = \phi'(x) \pm \psi'(x) \pm \chi'(x) \pm \ldots$ &c.

60. *If* $f(x) = \phi(x)\,.\,\psi(x)$, *then* Rule IV. $f(x) = \phi(x)\psi(x)$.

$$f'(x) = \phi'(x)\,.\,\psi(x) + \phi(x)\,.\,\psi'(x).$$

For then $\dfrac{f(x') - f(x)}{x' - x}$

$$= \frac{\phi(x')\psi(x') - \phi(x)\psi(x') + \phi(x)\psi(x') - \phi(x)\psi(x)}{x' - x}$$

$$= \frac{\phi(x') - \phi(x)}{x' - x}\,\psi(x') + \phi(x)\,.\,\frac{\psi(x') - \psi(x)}{x' - x};$$

now the limiting value of the second member of this equation when x' approaches x is $\phi'(x)\psi(x) + \phi(x)\psi'(x)$, by Lemma VIII.;

hence, $f'(x) = \phi'(x)\psi(x) + \phi(x)\psi'(x)$.

Cor. 1. $f(x) = \frac{\phi(x)}{\psi(x)}$.

61. *If* $f(x) = \frac{\phi(x)}{\psi(x)}$, *then*

$$f'(x) = \frac{\phi'(x)\,\psi(x) - \phi(x)\,\psi'(x)}{\{\psi(x)\}^2}.$$

For then $f(x)\,\psi(x) = \phi(x)$; and, as in the rule just proved, we have

$$f'(x)\,\psi(x) + f(x)\,\psi'(x) = \phi'(x);$$

$$\text{whence } f'(x) = \frac{\phi'(x) - f(x)\,\psi'(x)}{\psi(x)}$$

$$= \frac{\phi'(x)\,\psi(x) - \phi(x)\,\psi'(x)}{\{\psi(x)\}^2},$$

putting for $f(x)$ its value.

Rule V. $f(x) = x^n$.

62. *If* $f(x) = x^n$, *then* $f'(x) = nx^{n-1}$; *whether* n *be integral or fractional, positive or negative.*

For then $\frac{f(x') - f(x)}{x' - x} = \frac{x'^n - x^n}{x' - x}$

$$= x^{n-1} \cdot \frac{z^n - 1}{z - 1} \text{ if we put } x' = xz.$$

Now $x' = x$ when $z = 1$, therefore by Lemma II. the limiting value of the second member of this equation when z approaches 1 is the same thing as that of the first member when x' approaches x; but by Lemma XI. n is the limiting value of $\frac{z^n - 1}{z - 1}$ when z approaches 1. Hence $f'(x) = nx^{n-1}$.

Examples of the use of these five rules.

63. By means of these five Rules we may find the derivative of any rational function of x. For example:

Let $f(x) = a_0 + a_1 x + a_2 x^2 \ldots\ldots a_n x^n$,

then by Rule III. $f'(x)$ is found by taking the derivative of each term separately; also by Rule I. the derivative of a_0 is 0, and by Rule II. the derivative of any other term $a_m x^m$ is $a_m \times$ the derivative of x^m, or $a_m m x^{m-1}$ by Rule V.

Hence $\quad f'(x) = a_1 + 2a_2x + 3a_3x^2 \ldots\ldots na_nx^{n-1}$.

Again, let $\quad f(x) = \dfrac{x^2 - a^2}{x^2 + a^2} = \dfrac{\phi(x)}{\psi(x)}$ suppose;

then $\phi'(x) = 2x$, $\psi'(x) = 2x$, and therefore by Cor. Rule IV.

$$f'(x) = \frac{2x(x^2 + a^2) - (x^2 - a^2)\,2x}{(x^2 + a^2)^2} = \frac{4a^2x}{(x^2 + a^2)^2}.$$

Again, let $\quad f(x) = Ax^{\frac{5}{2}} + Bx^{-3}$,

then $\quad f'(x) = \frac{5}{2}Ax^{\frac{3}{2}} - 3Bx^{-4}$ by Rules II. III. V.

Again, let

$$f(x) = (x^2 + x^{-2})(x^{\frac{1}{2}} + x^{-\frac{1}{2}}) = \phi(x) \cdot \psi(x) \text{ suppose;}$$

then $\quad \phi'(x) = 2x - 2x^{-3}, \quad \psi'(x) = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$;

hence by Rule IV.

$$f'(x) = 2(x - x^{-3})(x^{\frac{1}{2}} + x^{-\frac{1}{2}}) + \tfrac{1}{2}(x^2 + x^{-2})(x^{-\frac{1}{2}} - x^{-\frac{3}{2}}).$$

(For more examples see Appendix A.)

64. *If* $f(x) = \log_a x$, *then* $f'(x) = \dfrac{1}{\log a} \cdot \dfrac{1}{x}$. Rule VI. $f(x)=\log_a x$.

For then

$$\frac{f(x') - f(x)}{x' - x} = \frac{\log_a x' - \log_a x}{x' - x}$$

$$= \frac{1}{x}\,\frac{\log_a(xz) - \log_a x}{z - 1} \text{ if we put } x' = xz$$

$$= \frac{1}{x}\,\frac{\log_a z}{z - 1}.$$

Now $x' = x$ when $z = 1$, therefore, by Lemma II., and Lemma XIII. Cor. 2, $\dfrac{1}{\log a} \cdot \dfrac{1}{x}$ is the limiting value of $\dfrac{f(x') - f(x)}{x' - x}$ when x' approaches x, i.e. $f'(x) = \dfrac{1}{\log a} \cdot \dfrac{1}{x}$.

3

Cor. $f(x) = \log x$. If $a = e$, then $\log a = 1$, which gives $f'(x) = \frac{1}{x}$.

Reason why e is usually taken as the base of logarithms. It appears therefore that we get a very simple expression for the derivative of the logarithm of x when e is taken as the base. It is for this reason that this base is chosen in all analytical investigations, except when numerical calculations come in, in which case 10 is a much more convenient base.

Rule VII. $f(x) = a^x$. 65. *If* $f(x) = a^x$, *then* $f'(x) = \log a \,.\, a^x$.

For then

$$\frac{f(x') - f(x)}{x' - x} = \frac{a^{x'} - a^x}{x' - x}$$

$$= a^x \,.\, \frac{a^z - 1}{z} \text{ if we put } x' - x = z.$$

Now $x' = x$ when $z = 1$, therefore, by Lemma II., and by Lemma XIII. Cor. 3, $a^x \log a$ limiting value of $\frac{f(x') - f(x)}{x' - x}$ when x' approaches x. Hence $f'(x) = \log a \,.\, a^x$.

Cor. $f(x) = e^x$. If $f(x) = e^x$, then $f'(x) = e^x$, since $\log_e e = 1$.

Rule VIII. $f(x) = \sin x$ or $\cos x$. 66. *If* $f(x) = \sin x$, *then* $f'(x) = \cos x$; *and if* $f(x) = \cos x$, *then* $f'(x) = -\sin x$.

For if $f(x) = \sin x$

$$\frac{f(x') - f(x)}{x' - x} = \frac{\sin x' - \sin x}{x' - x}$$

$$= \frac{2 \cos \frac{x' + x}{2} \,.\, \sin \frac{x' - x}{2}}{x' - x}$$

$$= \cos(x + z) \frac{\sin z}{z} \text{ if we put } \frac{x' - x}{2} = z.$$

Now the limiting value of the second member of this equation when z approaches 0 is $\cos x$, by Lemma VIII. and

Lemma X.; therefore, since $x' = x$ when $z = 0$, we have by Lemma II. $f'(x) = \cos x$.

And if $f(x) = \cos x$ we find similarly

$$\frac{f(x') - f(x)}{x' - x} = -\frac{2 \sin \dfrac{x' + x}{2} \sin \dfrac{x' - x}{2}}{x' - x}$$

$$= - \sin (x + z) \sin z ;$$

and therefore $f'(x) = - \sin x$.

Hence if $f(x) = \operatorname{cosec} x = \dfrac{1}{\sin x}$, $f'(x) = - \dfrac{\cos x}{\sin^2 x}$, by Cor. $f(x) = \sec x$ or $\operatorname{cosec} x$.

Rule IV. Cor. 1, putting $\phi(x) = 1$ and $\psi(x) = \sin x$.

And if $f(x) = \sec x = \dfrac{1}{\cos x}$, $f'(x) = \dfrac{\sin x}{\cos^2 x}$ similarly.

67. *If* $f(x) = \tan x$, *then* $f'(x) = \dfrac{1}{\cos^2 x}$, *and if* $f(x) = \cot x$, Rule IX. $f(x) = \tan x$ or $\cot x$.

then $f'(x) = -\dfrac{1}{\sin^2 x}$.

For if $f(x) = \tan x$,

$$\frac{f(x') - f(x)}{x' - x} = \frac{\tan x' - \tan x}{x' - x}$$

$$= \frac{1}{\cos x' \cos x} \cdot \frac{\sin (x' - x)}{x' - x}$$

$$= \frac{1}{\cos (x + z) \cos x} \cdot \frac{\sin z}{z} \text{ if we put } x' - x = z.$$

Hence, as in the former case, we have $f'(x) = \dfrac{1}{\cos^2 x}$.

And if $f(x) = \cot x$,

$$\frac{f(x') - f(x)}{x' - x} = - \frac{1}{\sin x' \sin x} \cdot \frac{\sin (x' - x)}{x' - x} ;$$

and therefore $f'(x) = - \dfrac{1}{\sin^2 x}$.

Cor. This rule deducible from the preceding.

This Rule may easily be deduced from the preceding, as follows:

If $f(x) = \tan x = \dfrac{\sin x}{\cos x}$ then by Rule IV. Cor. and by Rule VIII,

$$f'(x) = \frac{\cos x \cos x - \sin x(-\cos x)}{\cos^2 x} = \frac{1}{\cos^2 x}$$

and if $f(x) = \cot x = \dfrac{\cos x}{\sin x}$,

$$\text{then } f'(x) = \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} = -\frac{1}{\sin^2 x}.$$

Rule X. $f(x)=\sin^{-1}x$ or $\cos^{-1}x$.

68. *If* $f(x) = \sin^{-1}x$, *then* $f'(x) = \dfrac{1}{\sqrt{1-x^2}}$, *and if* $f(x) = \cos^{-1}x$, *then* $f'(x) = -\dfrac{1}{\sqrt{1-x^2}}$.

For if $f(x) = \sin^{-1}x$, and therefore $x = \sin f(x)$, we have

$$\frac{f(x') - f(x)}{x' - x} = \frac{f(x') - f(x)}{\sin f(x') - \sin f(x)}$$

$$= \frac{f(x') - f(x)}{2\cos\dfrac{f(x') + f(x)}{2}\sin\dfrac{f(x') - f(x)}{2}}$$

$$= \frac{1}{\cos\{f(x) + z\}} \cdot \frac{z}{\sin z},$$

if we put $\dfrac{f(x') - f(x)}{2} = z$.

Now $x' = x$ when $z = 0$; hence, by Lemmas II. and X. we have

$$f'(x) = \frac{1}{\cos f(x)} = \frac{1}{\sqrt{1-x^2}}, \text{ since } x = \sin f(x).$$

And similarly, we may shew that if $f(x) = \cos^{-1}x$,

$$f'(x) = -\frac{1}{\sqrt{1-x^2}}.$$

69. *If* $f(x) = \tan^{-1} x$, *then* $f'(x) = \frac{1}{1+x^2}$, *and if* $f(x) = \cot^{-1} x$, *then* $f'(x) = -\frac{1}{1+x^2}$. Rule XI. $f(x)=\tan^{-1}x$ or $\cot^{-1}x$.

For if $f(x) = \tan^{-1} x$,

$$\frac{f(x') - f(x)}{x' - x} = \frac{f(x') - f(x)}{\tan f(x') - \tan f(x)}$$

$$= \cos f(x') \cos f(x) \cdot \frac{f(x') - f(x)}{\sin \{f(x') - f(x)\}}$$

$$= \cos \{f(x) + z\} \cos f(x) \cdot \frac{z}{\sin z},$$

if we put $f(x') - f(x) = z$.

Hence, as before,

$$f'(x) = \cos^2 f(x) = \frac{1}{1+x^2}, \text{ since } \tan f(x) = x.$$

And similarly, we may shew that if $f(x) = \cot^{-1}(x)$, $f'(x) = -\frac{1}{1+x^2}$.

We have proved Rules VII, X, XI, directly, but they may be deduced very simply from Rules VI, VIII, IX, by means of the following Rule, as we shall shew.

70. *If we are given* $x = \phi(y)$, *in which case* y *will be some function of* x, $f(x)$ *suppose; then* $f'(x) = \frac{1}{\phi'(y)}$. Rule XII. If $x=\phi(y)$ and thence $y=f(x)$ to find $f'(x)$.

For then $\frac{f(x') - f(x)}{x' - x} = \frac{y' - y}{\phi(y') - \phi(y)}$,

$$= \frac{1}{\frac{\phi(y') - \phi(y)}{y' - y}}.$$

Now, by Lemma VIII, the limiting value of the second member of this equation when y' approaches y is $\frac{1}{\phi'(y)}$, and $x' = x$ when $y' = y$; hence, by Lemma II,

$$f'(x) = \frac{1}{\phi'(y)}.$$

Rules VII. X. XI. deduced from Rules VI. VIII. IX. by means of Rule XII.

71. This Rule is of great use in finding the derivatives of inverse functions, and by means of it may deduce Rules VII, X, XI, from Rules VI, VIII, IX, respectively, as follows:

If $x = \log_a y = \phi(y)$ suppose, then $y = a^x = f(x)$ suppose; and therefore, since

$$f'(x) = \frac{1}{\phi'(y)}, \text{ and } \phi'(y) = \frac{1}{\log a}\frac{1}{y}, \text{ by Rule VI,}$$

we have $f'(x) = \log a \,.\, y = \log a\ a^x$; which is Rule VII.

If $x = \sin y = \phi(y)$ suppose, then $y = \sin^{-1} x = f(x)$ suppose; and therefore, since

$$f'(x) = \frac{1}{\phi'(y)} = \frac{1}{\cos y}, \text{ by Rule VIII,}$$

we have $f'(x) = \dfrac{1}{\sqrt{1-x^2}}$, since $\cos y = \sqrt{1-x^2}$; which is Rule X.

If $x = \tan y = \phi(y)$ suppose, then $y = \tan^{-1} x = f(x)$ suppose; and we have therefore

$$f'(x) = \frac{1}{\phi'(y)} = \cos^2 y, \text{ by Rule IX. which } = \frac{1}{1+x^2}, \text{ since}$$

$\tan y = x$; which is Rule XI.

Rule XII. applied to the case of $\sec^{-1} x$ $\text{vers}^{-1} x$.

72. We may also by this Rule obtain the following derivatives which it is sometimes useful to remember.

If $x = \sec y = \phi(y)$, then $y = \sec^{-1} x = f(x)$, and we have therefore

$$f'(x) = \frac{1}{\phi'(x)} = \frac{\cos^2 y}{\sin y}, \text{ by Cor. Rule VIII.} = \frac{1}{x\sqrt{x^2-1}},$$

since $\cos y = \dfrac{1}{x}$.

Hence, if $f(x) = \sec^{-1} x, \quad f'(x) = \dfrac{1}{x\sqrt{x^2-1}}$.

Again, if $x = \text{vers}\, y = \phi(y)$, then $y = \text{vers}^{-1} x = f(x)$, and we have therefore

$$f'(x) = \frac{1}{\phi'(y)} = \frac{1}{\sin y}, \text{ by Rule VIII, since } \operatorname{vers} y = 1 - \cos y$$

$$= \frac{1}{\sqrt{2x - x^2}}, \text{ since } \cos y = 1 - x.$$

Hence, if $f(x) = vers^{-1} x$, $f'(x) = \dfrac{1}{\sqrt{2x - x^2}}$.

73. *If we are given two relations between* x *and* y *by the intervention of a third variable* v *in the form* Rule XIII. If $y = \phi(v)$ $v = \psi(x)$, and thence $y = f(x)$ to find $f'(x)$.

$$y = \phi(v), \quad \textit{and} \quad v = \psi(x),$$

in which case y *will be some function of* x, $f(x)$ *suppose; then* $f'(x) = \phi'(v)\,\psi'(x)$.

For then we have

$$\frac{f(x') - f(x)}{x' - x} = \frac{\phi(v') - \phi(v)}{v' - v} \cdot \frac{\psi(x') - \psi(x)}{x' - x}.$$

Now, since $v' = v$ when $x' = x$, the limiting value of $\dfrac{\phi(v') - \phi(v)}{v' - v}$ when x' approaches x, is the same (by Lem. II.) as that of $\dfrac{\phi(v') - \phi(v)}{v' - v}$ when v' approaches v, which $= \phi'(v)$; hence, by Lem. VIII, the limiting value of the second member of this equation, when x' approaches x, is $\phi'(v)\,\psi'(x)$; and therefore $f'(x) = \phi'(v)\,\psi'(x)$.

74. In the same manner we may shew, that if $y = \phi(v)$, $v = \psi(u)$, $u = \chi(x)$, and therefore y some function of x, $f(x)$ suppose; then $f'(x) = \phi'(v)\,\psi'(u)\,\chi'(x)$. And a similar result if any number of variables intervene between y and x. Cor. Any number of variables intervening between y and x.

For as before we have

$$\frac{f(x') - f(x)}{x' - x} = \frac{\phi(v') - \phi(v)}{v' - v} \cdot \frac{\psi(u') - \psi(u)}{u' - u} \cdot \frac{\chi(x') - \chi(x)}{x' - x};$$

$$\text{and } \therefore f'(x) = \phi'(v) \cdot \psi'(u) \cdot \chi'(x).$$

Examples of the use of this Rule. **75.** This Rule is of great use in finding the derivatives of complex functions, as the following examples will shew.

$$\text{Let } f(x) = \phi(c + x), \quad \text{put } c + x = v = \psi(x),$$
$$\text{and } \therefore \psi'(x) = 1;$$
$$\text{then } f'(x) = \phi'(v) = \phi'(c + x).$$

Hence the derivative of $\phi(c + x)$ is $\phi'(c + x)$. Thus the derivatives of $(c + x)^n$, $\log(c + x)$, e^{c+x}, $\sin(c + x)$, &c., are respectively, $n(c + x)^{n-1}$, $\dfrac{1}{c + x}$, e^{c+x}, $\cos(c + x)$, &c., by Rules V, VI, VII, VIII.

$$\text{Let } f(x) = \phi(cx), \text{ put } cx = v = \psi(x), \text{ and } \therefore \psi'(x) = c,$$
$$\text{then } f'(x) = \phi'(v)\,c = c\phi'(cx).$$

Hence the derivative of $\phi(cx)$ is $c\phi'(cx)$. Thus the derivatives of e^{cx}, $\tan^{-1}\dfrac{x}{c}$, $\sin^{-1}\dfrac{x}{c}$, are respectively

$$ce^{cx}, \quad \frac{c}{c^2 + x^2}, \quad \frac{1}{\sqrt{c^2 - x^2}}, \text{ by Rules VII, X, XI.}$$

$$\text{Let } f(x) = \phi(x^n), \quad \text{put } x^n = v = \psi(x),$$
$$\text{and } \therefore \psi'(x) = ux^{n-1} \quad \text{(Rule V.)},$$
$$\text{then } f'(x) = \phi'(v)\,nx^{n-1} = n\,\phi'(x^n)\,x^{n-1}.$$

Hence the derivative of $\phi(x^n)$, is $n\,\phi'(x^n)\,x^{n-1}$.

$$\text{Let } f(x) = e^{e^x}, \quad \text{put } v = e^x = \psi(x),$$
$$\text{and } \therefore f(x) = e^v = \phi(v);$$
$$\text{then } f'(x) = \phi'(v)\,\psi'(x) = e^v e^x \text{ (Rule VII.)} = e^{e^x}.\,e^x = e^{x+e^x}.$$

$$\text{Let } f(x) = e^{x+e^x}, \quad \text{put } v = x + e^x = \psi(x),$$
$$\text{and } \therefore f(x) = e^v = \phi(v);$$
$$\text{then } f'(x) = \phi'(v)\,\psi'(x) = e^v(1 + e^x) \text{ (Rules VII, III.)}$$
$$= e^{e^x}(1 + e^x).$$

Let $f(x) = \log \psi(x)$, put $v = \psi(x)$,

and $\therefore f(x) = \log v = \phi(v)$;

$$\text{then} \quad f'(x) = \phi'(v)\,\psi'(x) = \frac{1}{v}\,\psi'(x) = \frac{\psi'(x)}{\psi(x)}.$$

(For more Examples see Appendix B.)

76. *The derivative of a complicated product may often be conveniently found by taking its logarithm first.* Rule XIV. The derivative of a complicated product found by logarithms.

Let $f(x) = \{\phi(x)\}^m \cdot \{\psi(x)\}^n \cdot \{\chi(x)\}^p \ldots$

then $\log f(x) = m \log \phi(x) + n \log \psi(x) + p \log \chi(x) \ldots$

hence, taking the derivative by Rules XIII, III, and II,

$$\frac{f'(x)}{f(x)} = m\frac{\phi'(x)}{\phi(x)} + n\frac{\psi'(x)}{\psi(x)} + p\frac{\chi'(x)}{\chi(x)} \ldots$$

which gives $f'(x)$ very readily, as the following example will shew.

Let $f(x) = e^x \sqrt{\dfrac{1+x}{1-x}}$; Example.

then $\log f(x) = x + \frac{1}{2}\log(1+x) - \frac{1}{2}\log(1-x)$;

$$\therefore \frac{f'(x)}{f(x)} = 1 + \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right) = 1 + \frac{1}{1-x^2} = \frac{2-x^2}{1-x^2};$$

$$\therefore f'(x) = e^x \sqrt{\frac{1+x}{1-x}} \cdot \frac{2-x^2}{1-x^2}.$$

Let $f(x) = \dfrac{\sin x \cdot \sqrt{1-x^2}}{x^2}$;

then $\log f(x) = \log \sin x + \frac{1}{2}\log(1-x^2) - 2\log x$;

$$\therefore \frac{f'(x)}{f(x)} = \cot x + \frac{-x}{1-x^2} - \frac{2}{x} = \cot x - \frac{2-x^2}{x(1-x^2)};$$

$$\therefore f'(x) = \frac{\sin x\sqrt{1-x^2}}{x^2}\left\{\cot x - \frac{2-x^4}{x(1-x^2)}\right\}.$$

Let $f(x) = x^x$; then $\log f(x) = x \log x$,

and therefore we have

$$\frac{f'(x)}{f(x)} = \log x + x \cdot \frac{1}{x};$$

$$\therefore \; f'(x) = x^x (\log x + 1).$$

†† 77. * To these Rules we must add two more, which will be found of great use in many cases, and important hereafter, when we come to consider more than one independant variable. But we must first explain a certain notation which we shall make use of.

Partial derivatives; what. The notation $\phi'(\underset{\cdot}{x}y)$ and $\phi'(x\underset{\cdot}{y})$

If $f(xy)$ be any function of two variables x and y, we shall assume $f'(\underset{\cdot}{x}y)$ to denote the limiting value of $\dfrac{f(x'y) - f(xy)}{x' - x}$ when x' approaches x. It is evident that this limiting value is the derived function of $f(xy)$, taken on the supposition that y is constant and x alone variable. It is called the *Partial Derivative* of $f(xy)$ with respect to x.

In the same manner we shall take $f'(x\underset{\cdot}{y})$ to denote the limiting value of $\dfrac{f(xy') - f(xy)}{y' - y}$, when y' approaches y; i. e. *the Partial Derivative* of $f(xy)$ with respect to y, or the derivative taken on the supposition that y alone is variable.

Examples.

Thus if $f(xy) = x^2 + xy + y^2$, then

$$f'(\underset{\cdot}{x}y) = 2x + y, \text{ and } f'(x\underset{\cdot}{y}) = y + 2x.$$

Again, if $f(xy) = x \sin y + ye^x$,

$$f'(\underset{\cdot}{x}y) = \sin y + ye^x, \text{ and } f'(x\underset{\cdot}{y}) = x \cos y + e^x.$$

Thus by dashing the functional letter we denote a derivative of a function of two variables, and by a dot under one of the variables we signify that *that* variable alone is supposed to vary.

This notation does not necessarily require that

In using this notation we do not necessarily assume that y is independant of x; for whether y be independ-

* The Articles marked thus †† may be omitted till they are referred to, at least on a first perusal

ant of x or not, we may always put x' in place of x if we please, and so find $f(x'y)$, and thence $\dfrac{f(x'y)-f(xy)}{x'-x}$; and then, by the previous Rules, we may determine the limiting value of this quantity when x' approaches x. And the same may be said of $f'(xy)$. The following is the Rule for which it is necessary to make use of this notation or something equivalent to it. x and y should be independant of each other.

†† 78. *If we are given a relation between* y *and two other variables* u *and* v *which are functions of* x ; *i. e. if* Rule XV.

$$y = \phi(uv), \quad u = \psi(x), \quad v = \chi(x),$$

in which case y *will be some function of* x, f(x) *suppose; then*

$$f'(x) = \phi'(u\,v)\,\psi'(x) + \phi'(u\,v)\,\chi'(x).$$

For $\dfrac{f(x')-f(x)}{x'-x} = \dfrac{\phi(u'v')-\phi(uv')}{x'-x} + \dfrac{\phi(uv')-\phi(uv)}{x'-x}$,

and $\dfrac{\phi(u'v')-\phi(uv')}{x'-x} = \dfrac{\phi(u'v')-\phi(uv')}{u'-u} \cdot \dfrac{\psi(x')-\psi(x)}{x'-x}$

$$\frac{\phi(uv')-\phi(uv)}{x'-x} = \frac{\phi(uv')-\phi(uv)}{v'-v} \cdot \frac{\chi(x')-\chi(x)}{x'-x}.$$

Now when x' approaches x, the limiting values of

$$\frac{\psi(x')-\psi(x)}{x'-x}, \quad \frac{\phi(uv')-\phi(uv)}{v'-v}, \text{ and } \frac{\chi(x')-\chi(x)}{x'-x},$$

are respectively $\psi'(x)$, $\phi'(uv)$, and $\chi'(x)$. Also since v' becomes v when $x' = x$, the limiting value of $\dfrac{\phi(u'v')-\phi(uv')}{u'-u}$*

* If we suppose for a moment that v' is not $\chi(x')$ but some constant, then $\phi'(uv')$ is the limiting value of $\dfrac{\phi(u'v')-\phi(uv')}{u'-u}$ when x' approaches x therefore $\dfrac{\phi(u'v')-\phi(uv')}{u'-u} \sim \phi'(uv')$ may be diminished *ad libitum* by sufficiently diminishing $x' \sim x$; and this is evidently true whatever be the value of v'; therefore it must be true if we suppose v' to vary in any manner while we diminish $x' \sim x$; it is therefore true if $v' = \chi(x')$, and therefore by Lemma VI. Cor. 2, $\phi'(uv)$, which is the limiting value of $\phi'(uv')$, is also that of $\dfrac{\phi(u'v')-\phi(uv')}{u-u}$, supposing $v' = \chi(x')$.

is evidently the same thing as that of $\dfrac{\phi(u'v) - \phi(uv)}{u' - u}$, which is $\phi'(\underset{\cdot}{u}v)$. Hence we have

$$f'(x) = \phi'(\underset{\cdot}{u}v)\,\psi'(x) + \phi'(u\underset{\cdot}{v})\,\chi'(x).$$

Examples.

Let $\phi(uv) = u^v$,

then $\phi'(\underset{\cdot}{u}v) = vu^{v-1}$ and $\phi'(u\underset{\cdot}{v}) = \log u\, u^v$;

$$\therefore\ f'(x) = vu^{v-1}\psi'(x) + \log u\, u^v \chi'(x).$$

Suppose that $u = x$ and $v = x$, and $\therefore\ \psi'(x) = 1\ \chi'(x) = 1$; then $f'(x) = x^x + \log x\, x^x = x^x(1 + \log x)$.

Again, let $\phi(uv) = u^2 + v^2 - uv$;

then $\phi'(\underset{\cdot}{u}v) = 2u - v$, $\phi'(u\underset{\cdot}{v}) = 2v - u$;

$$\therefore\ f'(x) = (2u - v)\,\psi'(x) + (2v - u)\,\chi'(x).$$

Cor. If $y = \phi(uvw)$ w being another function of x, $\zeta(x)$, suppose then by putting $f(x') - f(x)$ in the form

$$\{\phi(u'v'w') - \phi(uv'w')\} + \{\phi(uv'w') - \phi(uvw')\}$$
$$+ \{\phi(uvw') - \phi(uvw)\}.$$

We may shew exactly as before, that

$$f'(x) = \phi'(\underset{\cdot}{u}vw)\,\psi'(x) + \phi'(u\underset{\cdot}{v}w)\,\chi'(x) + \phi'(uv\underset{\cdot}{w})\,\zeta'(x).$$

This Rule we may evidently extend to the case where y is a function of any number of functions of x.

Rule XVI. ††79. *If an equation be given between* x *and* y, *which of course makes* y *a function of* x, *we may find the derivative of* y *by means of Rule* XV.

For, let the given equation be

$$\phi(yx) = 0;$$

in virtue of this equation $y =$ some function of x, $\psi(x)$ suppose; and $\phi(yx)$, by substituting for y this value, becomes also a function of x, $f(x)$ suppose: then, by Rule XV, putting y for u, and v for x, and therefore $\chi'(x) = 1$, we have

$$f(x) = \phi'(\underset{\cdot}{y}x)\,\psi'(x) + \phi'(y\underset{\cdot}{x}).$$

Now, $f(x) = 0$, and therefore by Rule I, $f'(x) = 0$; therefore we have

$$\phi'(\underset{\cdot}{y}x)\,\psi'(x) + \phi'(y\underset{\cdot}{x}) = 0\,;$$

$$\text{or,} \quad \psi'(x) = -\frac{\phi'(y\underset{\cdot}{x})}{\phi'(\underset{\cdot}{y}x)}\,.$$

Let the given equation be Example.

$$x^2 + y^2 - axy + b = 0,$$

$$\text{here,} \quad \phi'(\underset{\cdot}{y}x) = 2y - ax, \quad \phi'(y\underset{\cdot}{x}) = 2x - ay;$$

$$\therefore \text{ if } \psi(x) = y, \quad (2y - ax)\,\psi'(x) + 2x - ay = 0\,;$$

$$\text{or,} \quad \psi'(x) = -\frac{2x - ay}{2y - ax}\,.$$

CHAPTER VI.

THE DIFFERENTIAL NOTATION.

80. We have hitherto adopted Lagrange's notation, $f'(x)$, to represent the limiting value of $\dfrac{f(x')-f(x)}{x'-x}$ when x' approaches x; this notation is often very convenient, and is perhaps more easily understood than any other, at least when the student enters upon this subject for the first time; and this is the reason why we have used it in obtaining the preceding Rules. But there is a far more elegant and powerful notation, due to Leibnitz, called the differential notation, which we now proceed to explain; not however for the purpose of abandoning the former notation, for we shall often make use of it, as it is preferable in certain cases to any other.

The ratio of the symbols $df(x)$ and dx taken to represent $f'(x)$.

81. In the differential notation instead of representing the limiting ratio of $\dfrac{f(x')-f(x)}{x'-x}$ when x' approaches x, by a single quantity, such as $f'(x)$, we represent it by the ratio of two arbitrary quantities denoted by the symbols $df(x)$ and dx, which, for a reason we shall explain, are called the *differentials* of $f(x)$ and x; the letter d being simply an abbreviation of the words "*differential of.*"

Definition of $df(x)$ and dx.

We define $df(x)$ and dx as follows:

$\mathrm{df(x)}$ *and* dx *are two quantities whose ratio* $\dfrac{\mathrm{df(x)}}{\mathrm{dx}}$ *is equal to the limiting value of* $\dfrac{\mathrm{f(x')-f(x)}}{\mathrm{x'-x}}$ *when* $\mathrm{x'}$ *approaches* x. *Or more simply;* $\mathrm{df(x)}$ *and* dx *are two quantities such that* $\dfrac{\mathrm{df(x)}}{\mathrm{dx}} = \mathrm{f'(x)}$.

In this definition, all that we say of $df(x)$ and dx is this, "*that they are in a certain ratio;*" therefore they are arbitrary quantities, *and we may give to either of them any value we please, constant or variable*, provided we give to the other that value which makes $\dfrac{df(x)}{dx} = f'(x)$. One of the quantities $df(x)$, dx is quite arbitrary.

82. It may be asked respecting this notation, is there not some degree of vagueness in representing a definite quantity $f'(x)$ by the ratio of two arbitrary quantities? Objection to this notation answered.

To this we may answer, that we often do the very same thing in common Algebra without any vagueness; we often say let $\dfrac{m}{n}$ represent such and such a quantity, instead of let m represent such a quantity. Thus, suppose it required to divide a quantity a into two parts in a given ratio: we might proceed thus. Let m and n be any two quantities which are in the given ratio, and let x and y be the two parts, then $x + y = a$, and $\dfrac{x}{y} = \dfrac{m}{n}$;

$$\therefore \left(\frac{m}{n} + 1\right) y = a, \text{ or, } y = \frac{na}{m+n}; \text{ and } \therefore x = \frac{ma}{m+n}.$$

Or we might proceed thus, let m be the given ratio, then

$$\frac{x}{y} = m, \quad (m+1)y = a; \text{ and } \therefore y = \frac{a}{m+1} \quad x = \frac{ma}{m+1}.$$

It is evident that there is not the least degree of vagueness here in representing the given ratio by $\dfrac{m}{n}$, instead of simply by m*,

* The chief practical advantages we gain by representing $f'(x)$ by the ratio $\dfrac{df(x)}{dx}$ seem to me to be these. We may often suppose $df(x)$ or dx to have some value which will simplify our expressions (as will appear in changing the independant variable hereafter.) The application of Rule XIII. to complex functions is very much facilitated by using the notation $\dfrac{df(x)}{dx}$. We may often advantageously preserve the symmetry of our expressions by using this notation. It is almost Advantages of the notation $\dfrac{df(x)}{dx}$.

Origin of the notation $\frac{df(x)}{dx}$, and of the term differential.

83. Again it may be asked, what is the reason of using the letter d before $f(x)$ and x, to represent the terms of the ratio by which we denote $f'(x)$? This question may be answered by the following very brief account of the origin of this notation and of the term differential.

$f(x') - f(x)$, and $x' - x$ are the differences between corresponding values of $f(x)$ and x, and they are often represented by a δ prefixed to $f(x)$ and x, in this manner, viz. $\delta f(x)$, δx; δ being simply an abbreviation of the words "*the difference between two values of.*"

Now, $\delta f(x)$ and δx become zero when $x' = x$, and then their ratio $\frac{\delta f(x)}{\delta x}$ ceases to be a definite quantity; but so long as x' is not actually equal to x the ratio $\frac{\delta f(x)}{\delta x}$ is a definite quantity. We may therefore conceive $\delta f(x)$ and δx as small as we please though not actually zero, without rendering our conception of the ratio $\frac{\delta f(x)}{\delta x}$ at all vague or difficult: indeed it is just as easy to conceive that a definite ratio subsists between $\delta f(x)$ and δx when they are in a state of extreme smallness as when they are of ordinary magnitude for our idea of a ratio is quite independent of the *actual* magnitude of the quantities composing it.

The differences $\delta f(x)$ and δx when in a state of extreme smallness were called *differentials* by Leibnitz (i.e. minute differences), and the symbols $df(x)$ and dx were made use of by him to represent them; d, like δ, being simply an abbreviation of the words "*differential of.*"

In all calculations into which these differentials entered he supposed them to be what are called *infinitesimals*, i.e. quantities so small, that they may without error be neglected compared with ordinary quantities, and on this sup-

impossible to represent what are called total differentials without this notation. It is peculiarly adapted to the case of definite and multiple integrals in the Integral Calculus. And it is a very expressive notation, which makes it peculiarly convenient in mixed mathematics; e g. in the case of the principle of virtual velocities applied to an example

position he investigated rules for finding $df(x)$ in all cases, and made various interesting applications of these rules. In this manner it was that the differential calculus came into existence.

84. In order to explain more distinctly in what this method of Liebnitz consists, we shall apply it to the example given in (23). Example of the method of Leibnitz.

According to Leibnitz, we may conceive the triangle PQO (fig. 6) to become so extremely small, that the line PQ shall coincide in direction with the tangent at P; in which case the sides of the triangle OPQ will become infinitesimal quantities; then OQ or $f(x') - f(x)$ will be represented by $df(x)$ and PO by dx, and we shall have

$$\tan QPO \text{ or } \tan PTX = \frac{QO}{PO} = \frac{df(x)}{dx}.$$

Now let $\quad f(x) = \dfrac{x^2}{4m}$,

$$\text{then } f(x') - f(x) \text{ or } df(x) = \frac{x'^2 - x^2}{4m} = \frac{(x' + x)(x' - x)}{4m}$$
$$= \frac{(2x + dx)\,dx}{4m},$$

putting $x + dx$ for x'.

Now, according to the supposition that differentials are to be neglected compared with ordinary quantities, we must consider $2x + dx$ to be the same thing as $2x$, and therefore we have

$$df(x) = \frac{x}{2m}dx, \text{ and } \therefore \tan PTX = \frac{x}{2m},$$

which is the same result we obtained before.

85. Whether this is a strictly logical way of proceeding may be fairly questioned, though the conclusion arrived at is true: for it is easy to see that the correctness of the result arises from the compensation of two errors, namely, the erroneous supposition that the line PQ is coincident with the tangent at P, which it never can be so long as the Where this method is faulty. Reason why the result is correct.

triangle POQ has any existence, and the erroneous supposition that $2x + dx$ is equal to $2x$.

We shall endeavour however to shew hereafter, by means of the principles already established, that this method of using differentials *must* always lead to correct results, certain precautions being taken; and this is important to shew, since this method is extensively used, and indeed *must* be used in many cases to avoid complexity. At present we shall say no more upon this subject, except just to remark, that this is the manner in which the notation $\frac{df(x)}{dx}$ has come to represent the limiting value of $\frac{f(x') - f(x)}{x' - x}$ when x' approaches x: for it is easy to see, by the example just given, that $\frac{df(x)}{dx}$, simplified by neglecting the differentials $df(x)$ and dx compared with ordinary quantities, will always be the same thing as this limiting value.

We adopt the notation $\frac{df(x)}{dx}$ and the term differential, but not the notion that $df(x)$ and dx are infinitesimals.

86. It is important to observe here, that we adopt *only* the *notation* $\frac{df(x)}{dx}$ and the *term Differential*. We do not define $df(x)$ and dx to be what $\delta f(x)$ and δx become when in a state of extreme smallness; all that we say is this, that $df(x)$ and dx are two quantities, be they small or large, whose ratio is equal to $f'(x)$, i.e. the limiting value of $\frac{\delta f(x)}{\delta x}$ or $\frac{f(x') - f(x)}{x' - x}$ when x' approaches x.

Geometrical representation of the quantities $df(x)$ and dx.

87. In the example just considered, if we produce PO to any point U and draw US perpendicular to PU to meet SPT in S; then

$$\frac{SU}{PU} = \tan PTS = \text{lim. val. of} \frac{f(x') - f(x)}{x' - x} \text{ when } x' \text{ app. } x,$$
$$= f'(x);$$

hence SU and PU are the differentials of $f(x)$ and x, since they are two quantities whose ratio $= f'(x)$; we have therefore $SU = df(x)$, $PU = dx$.

88. Since $\frac{df(x)}{dx}$ and $f'(x)$ represent the same thing, we have How $f'(x)$ came to be called the differential coefficient.

$$\frac{df(x)}{dx} = f'(x), \text{ and } \therefore \; df(x) = f'(x)\,dx,$$

$f'(x)$ is therefore the coefficient by which we must multiply the differential of x to obtain the differential of $f(x)$; on this account $f'(x)$ has acquired the name of "*the differential coefficient*," and it is in this way that $\frac{df(x)}{dx}$, which $=f'(x)$, has come by the same name.

The obtaining of the relation between dy and dx is called *Differentiation*.

89. We may express the Rules Chap. V. in the differential notation by simply putting The rules obtained in last chapter stated in the differential notation.

$$\frac{df(x)}{dx}, \quad \frac{d\phi(x)}{dx}, \quad \frac{d\psi(x)}{dx}, \text{ \&c....}$$

for $f'(x)$, $\phi'(x)$, $\psi'(x)$, &c. respectively, and multiplying up dx. For the sake of neatness we shall put y, u, v, for $f(x)$, $\phi(x)$, $\psi(x)$, respectively: and therefore dy, du, dv, &c.... for $f'(x)\,dx$, $\phi'(x)\,dx$, $\psi'(x)\,dx$...&c. In this manner we have

Rule I. If $y =$ constant, then $dy = 0$.

Rule II. If $y = cu$, $dy = cdu$.

Rule III. If $y = u + v + w...$, $dy = du + dv + dw...$

Rule IV. If $y = uv$, $dy = vdu + udv$.

If $y = \frac{u}{v}$, $dy = \frac{vdu - udv}{v^2}$.

Rule V. If $y = x^n$, $dy = nx^{n-1}dx$.

Rule VI. If $y = \log x$, $dy = \dfrac{dx}{x}$.

Rule VII. If $y = a^x$, $dy = \log a . a^x dx$.

Rule VIII. If $y = \sin x$, $dy = \cos x dx$.

If $y = \cos x$, $dy = -\sin x dx$.

Rule IX. If $y = \tan x$, $dy = \dfrac{dx}{\cos^2 x}$.

If $y = \cot x$, $dy = -\dfrac{dx}{\sin^2 x}$.

Rule X. If $y = \sin^{-1} x$, $dy = \dfrac{dx}{\sqrt{1 - x^2}}$.

If $y = \cos^{-1} x$, $dy = -\dfrac{dx}{\sqrt{1 - x^2}}$.

Rule XI. If $y = \tan^{-1} x$, $dy = \dfrac{dx}{1 + x^2}$.

If $y = \cot^{-1} x$, $dy = -\dfrac{dx}{1 + x^2}$.

Rule XII. 90. Rule XII. shews that if we obtain dx in terms of dy from the equation $x = \phi(y)$, the value of $\dfrac{dy}{dx}$ so obtained is the proper value; i. e. the same value that would be obtained if we found y in terms of x directly, and then differentiated. For by differentiating the equation $x = \phi(y)$ as it stands, we find

$$dx = \phi'(y) dy, \quad \text{and } \therefore \frac{dy}{dx} = \frac{1}{\phi'(y)}.$$

Now if $f(x)$ be the value of y found directly in terms of x, we have $\dfrac{dy}{dx} = f'(x)$; which is the same as the former value, since

$f'(x) = \frac{1}{\phi'(y)}$ by Rule XII. Hence, whichever way we proceed, the value of $\frac{dy}{dx}$ obtained is the same.

91. Rule XIII. shews that if we find dy in terms of dv from the equation $y = \phi(v)$, and dv in terms of dx from the equation $v = \psi(x)$, and so dy in terms of dx; the value of $\frac{dy}{dx}$ thus found is the proper value; i. e. the same value that would be obtained if we found y in terms of x directly by eliminating v, and then differentiated. For by differentiating the equations $y = \phi(v)$, $v = \psi(x)$, we obtain Rule XIII.

$$dy = \phi'(v)\,dv, \quad dv = \psi'(x)\,dx, \quad \text{and } \therefore \frac{dy}{dx} = \phi'(v)\,\psi'(x).$$

Now if $f(x)$ be the value of y found directly in terms of x, we have $\frac{dy}{dx} = f'(x)$, which is the same as the former value, since $f'(x) = \phi'(v)\,\psi'(x)$ by Rule XIII. Hence, whichever way we proceed, the value of $\frac{dy}{dx}$ obtained is the same.

92. If we use the differential notation, this Rule does not appear to want any proof; for it is stated in that notation, as follows. If y be a function of v, and v a function of x, then $\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx}$. Now, since dv divides out in the second member, this appears to be self-evident. But we must remember that the $\frac{dy}{dx}$ in the first member is supposed to be derived from y expressed directly in terms of x; whereas the $\frac{dy}{dx}$ obtained from the second member, by dividing out dv, is not derived from y in this manner; and though it is easy to see that the result is the same in both cases, yet it ought to be formally proved for the sake of exactness. Rules XII and XIII seem to want no proof in the differential notation. Why nevertheless they ought to be proved.

The same may be said of the preceding Rule also.

For examples of the use of the differential notation, and of the Rules for differentiation in general, see Appendix (C).

Rule XV.

†† 93. To express Rule XV. in the differential notation, we shall assume $d_u y$ to denote the differential coefficient of y on the supposition that v is constant and u alone variable, and $d_v y$ to denote the differential coefficient of y on the supposition that u is constant and v alone variable; then we have

$$\phi'(\underset{\cdot}{u}v) = d_u y, \quad \phi'(u\underset{\cdot}{v}) = d_v y, \quad \psi'(x) = \frac{du}{dx}, \quad \chi'(x) = \frac{dv}{dx},$$

$$\text{and} \quad f'(x) = \frac{dy}{dx}, \text{ and therefore}$$

$$\frac{dy}{dx} = d_u y \frac{du}{dx} + d_v y \frac{dv}{dx};$$

$$\text{or} \quad dy = d_u y\, du + d_v y\, dv.$$

Hence Rule XV. may be stated thus.

If y be a function of two other functions of x u and v, then

$$dy = d_u y\, du + d_v y\, dv,$$

Differential of a function equal the sum of its partial differentials.

$d_u y\, du$ and $d_v y\, dv$ may be called the partial differentials of y with respect to u and v respectively. *Hence the differential of* y *is equal to the sum of its partial differentials with respect to* u *and* v *respectively.*

More generally, if y be a function of any number of functions of x, viz. $u, v, w\ldots$, we have by Cor. to Rule XV.

$$dy = d_u y\, du + d_v y\, dv + d_w y\, dw \ldots\ldots$$

The manner in which we arrive at the conception of the differential of a function

†† 94. Suppose that y is a function of $u, v, w, \ldots$ where $u, v, w, \ldots$ are variables to which we may assign any values we please independently of each other; then we may suppose $u, v, w, \ldots$ to be any arbitrary functions of a new variable x; for by altering the nature of these functions we may make

$u, v, w, \ldots$ vary in any manner we please when x varies, and so receive any arbitrary values: in fact a function of $u, v, w, \ldots$ where $u, v, w, \ldots$ are any arbitrary functions of x, is just the same thing as a function of $u, v, w, \ldots$ where $u, v, w, \ldots$ are any arbitrary quantities. The advantage of looking upon $u, v, w, \ldots$ as arbitrary functions of x, instead of mere arbitrary quantities, is this, that we thereby arrive very simply at the conception of the differential of a function of several independant variables: for, if y be a function of the independant variables $u, v, w, \ldots$, we may, if we choose, consider $u, v, w, \ldots$ to be arbitrary functions of a new variable x, and then by the result of the previous article we have

of several *independant* variables.

$$dy = d_u y\, du + d_v y\, dv + d_w y\, dw + \ldots\ldots$$

Thus if y *be a function of several independant variables, the differential of* y*, regarded in this manner, is the sum of the partial differentials of* y *with respect to each of the variables.*

What the quantities $du, dv, dw \ldots$ are depends upon what functions we suppose $u, v, w \ldots$ to be of x, and what value we give to dx; if we suppose $u = \phi(x)$, $v = \psi(x)$, $w = \chi(x)$, then $du = \phi'(x)\, dx$, $dv = \psi'(x)\, dx$, $dw = \chi'(x)\, dx \ldots$; and hence since ϕ, ψ, χ may be any functions whatever, and dx any quantity, it is clear that $du, dv, dw \ldots$ may have any values whatever we please to give them. We shall in general suppose that $u = Ax$, $v = Bx$, $w = Cx \ldots$ where $A, B, C \ldots$ are any arbitrary constants, and that dx is constant; and then $du, dv, dw \ldots$ will be arbitrary constants.

du, dv, dw, generally supposed to be arbitrary constants.

We shall therefore assume that if y be a function of several independant variables $u, v, w \ldots$ then the differential of y, which we shall denote by dy, is equal to the expression

Definition of the differential of a function of several independant variables.

$$d_u y \,.\, du + d_v y \,.\, dv + d_w y \,.\, dw \ldots$$

where du, dv, dw, &c. are in general any arbitrary constants; but may also receive any variable values we please to give them.

This assumption is really our definition of a differential of several *independant* variables; which definition the pecu-

liar form assumed by the differential of a function of several *dependant* variables suggests to us in the manner we have explained.

A total differential, what. dy here is often called the *Total Differential* of y, in contradistinction to the partial differentials $d_u y\, du$, $d_v y\, du$, &c.

Examples. Let $y = u^3 + v^3 + w^3 - 3uvw$,

$$\text{then} \quad d_u y = 3(u^2 - vw),$$

$$d_v y = 3(v^2 - uw)$$

$$d_w y = 3(w^2 - uv);$$

$$\text{and} \quad \therefore\ dy = 3\{(u^2 - vw)\,du + (v^2 - uw)\,dv + (w^2 - uv)\,dw\}.$$

Let $y = u^{vw}$,

$$\text{then} \quad dy = u^{vw-1}\{vw\,du + u\log u\,(v\,dw + w\,dv)\}.$$

Rule XVI. †† 95. Rule XVI. is thus expressed in the differential notation.

If U be a function of x and y, and we put $U = 0$, then

$$d_x U \,.\, dx + d_y U \,.\, dy = 0,$$

from which equation we immediately get $\dfrac{dy}{dx}$.

Example. Let the given equation be

$$U = \sin x \sin y - xy = 0,$$

$$\text{here} \quad d_x U = (\cos x \sin y - y), \quad d_y U = (\sin x \cos y - x);$$

$$\therefore\ (\cos x \sin y - y)\,dx + (\sin x \cos y - x)\,dy = 0,$$

$$\text{and} \quad \therefore\ \frac{dy}{dx} = -\frac{\cos x \sin y - y}{\sin x \cos y - x}.$$

CHAPTER VII.

SUCCESSIVE DERIVATIVES OR DIFFERENTIALS. CHANGE OF THE INDEPENDANT VARIABLE.

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96. In the same manner that we obtain the derivative $f'(x)$ of any function $f(x)$, so we may obtain the derivative of $f'(x)$, and again the derivative of that derivative, and so on. Thus if $f(x) = x^n$, $f'(x) = nx^{n-1}$, and the derivative of this is $n(n-1)x^{n-2}$; again the derivative of $n(n-1)x^{n-2}$ is $n(n-1)(n-2)x^{n-3}$, and so we may go on.

The n^{th} derivative or differential coefficient; what. We represent it by $f^n(x)$.

The derivative of $f'(x)$ is called the *Second Derivative* of $f(x)$, and is denoted by $f^2(x)$; the derivative of $f^2(x)$ is called the *Third Derivative* of $f(x)$, and is denoted by $f^3(x)$, and so on; and in general $f^n(x)$ denotes the

deriv. of [the deriv. of {the deriv.... (to n derivs.)}] of $f(x)$,

$f^n(x)$ is also called the n^{th} differential coefficient of $f(x)$.

$f^n(x)$ is represented by $\frac{d^n y}{dx^n}$ in the differential notation, on what hypothesis.

97. If we put $f(x) = y$, and therefore $f'(x) = \frac{dy}{dx}$, then

$$f^2(x) = \frac{df'(x)}{dx} = \frac{d\left(\frac{dy}{dx}\right)}{dx}.$$

Now we have seen in (80) that we may give to one of the quantities dy or dx any value we please, variable or constant; let us suppose dx a constant, then by Rule II,

$$d\left(\frac{dy}{dx}\right) = \frac{ddy}{dx}, \text{ and we have therefore } f^2(x) = \frac{ddy}{dx^2}.$$

$$\text{Again } f^3(x) = \frac{df^2(x)}{dx} = \frac{d\left(\frac{ddy}{dx^3}\right)}{dx} = \frac{dddy}{dx^3},$$

as before, and so on, and in general

$$f^n(x) = \frac{ddd\ldots(\text{to } n\ d\text{'s})\, y}{dx^n}.$$

To avoid the repetition of d's we represent d repeated n times by d^n, and thus we have

$$f^n(x) = \frac{d^n y}{dx^n}.$$

For examples of finding successive differential coefficients, see Appendix (D).

Expression for $f^n(x)$ when dx is not supposed constant.

98. We have supposed dx constant for the sake of simplicity; if we do not make this supposition, then we have

$$f^2(x) = \frac{d\left(\frac{dy}{dx}\right)}{dx},\ f^3(x) = \frac{df^2(x)}{dx} = \frac{d\left\{\frac{d\left(\frac{dy}{dx}\right)}{dx}\right\}}{dx} \ldots \text{\&c.} \ldots \text{\&c.}$$

which are the expressions for the successive derivatives of y considered as a function of x, whatever be the values of dx and dy. If we perform the differentiations, and put d^2 for dd, d^3 for ddd, &c., &c., we have

$$d\left(\frac{dy}{dx}\right) = \frac{d^2y\,dx - dy\,d^2x}{dx^2}, \text{ and } \therefore f^2(x) = \frac{d^2y\,dx - dy\,d^2x}{dx^3},$$

$$df^2(x) = \frac{(d^3y\,dx - dy\,d^3x)\,dx^3 - (d^2y\,dx - dy\,d^2x)\,3\,dx^2\,d^2x}{dx^6};$$

$$\therefore f^3(x) = \frac{(d^3y\,dx - dy\,d^3x)\,dx - 3\,(d^2y\,dx - dy\,d^2x)\,d^2x}{dx^5};$$

and so we may go on. The expressions however become extremely complicated.

In these expressions, if we suppose dx constant, we have

$$d^2x = 0, \quad d^3x = 0, \quad \text{\&c.} \ldots \text{ and therefore}$$

$$f^2(x) = \frac{d^2y}{dx^2}, \quad f^3(x) = \frac{d^3y}{dx^3}, \quad \text{\&c.},$$

which are of course the same expressions as in the preceding article.

If we suppose dy constant, we have $d^2y = 0$, $d^3y = 0$, and therefore

$$f^2(x) = -\frac{dy\,d^2x}{dx^3},\quad f^3(x) = -\frac{dy\,\{d^3x\,dx - 3(d^2x)^2\}}{dx^5}\ \dots$$

and so on.

99. That variable whose differential is supposed to be constant is called the independant variable; thus when we put $f^2(x) = \frac{d^2y}{dx^2}$, x is the independant variable; when we put $f^2(x) = -\frac{dy\,d^2x}{dx^3}$, y is the independant variable; and when we put $f^2(x) = \frac{d^2y\,dx - dy\,d^2x}{dx^3}$, neither x nor y is the independant variable. Independant variable, what.

100. It is important to remember that $\frac{d^ny}{dx^n}$ represents $f^n(x)$, *only* on the supposition that x is the independant variable, and therefore, though we may always put $d^ny = f^n(x)dx^n$ when x is supposed to be the independant variable, we may not do so if this be not supposed. d^ny, therefore, does not always represent the same quantity; what it represents depends upon what quantity is considered the independant variable. What d^ny represents depends upon what we consider to be the independant variable.

Thus suppose $y = x^2$, then $dy = 2x\,dx$, and $d^2y = 2dx^2$, if dx be constant. But suppose that $x = z^2$, and $\therefore y = z^4$; then $dy = 4z^3dz$, $d^2y = 12z^2dz^2$, if dz be constant; now since $x = z^2$, $dx = 2z\,dz$, and $\therefore 12z^2dz^2 = 3dx^2$, and $\therefore d^2y = 3dx^2$. Hence, when x is supposed the independant variable, $d^2y = 2dx^2$, and when z or $x^{\frac{1}{2}}$ is supposed the independant variable, $d^2y = 3dx^2$: from which it is clear that d^2y does not mean the same thing when x is the independant variable, and when $x^{\frac{1}{2}}$ is. Example.

101. It is often necessary to change the independant variable in an expression containing differentials; this we may

Change of the independant variable; how effected.

easily do by remembering always, 1st, that the independant variable is that variable whose differential is constant; 2ndly, that $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, &c. represent the successive derivatives of y considered as a function of x, on the express condition that dx is a constant; and 3rdly, that these derivatives, whatever be the independant variable, are represented by

$$\frac{dy}{dx}, \quad \frac{d\left(\frac{dy}{dx}\right)}{dx}, \quad \frac{d\left\{\frac{d\left(\frac{dy}{dx}\right)}{dx}\right\}}{dx} \ldots \text{\&c. \&c.} \ldots \text{ respectively.}$$

Hence, if we have any expression involving

$$\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \frac{d^3y}{dx^3} \ldots \text{\&c.}$$

where these quantities are supposed to represent the successive derivatives of y considered as a function of x, on which supposition x must necessarily be the independant variable: then since

$$\frac{dy}{dx}, \quad \frac{d\left(\frac{dy}{dx}\right)}{dx}, \quad \frac{d\left\{\frac{d\left(\frac{dy}{dx}\right)}{dx}\right\}}{dx} \ldots \text{\&c.}$$

represent the same derivatives, we may put these instead of

$$\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \frac{d^3y}{dx^3} \ldots \text{\&c. respectively;}$$

and then we may suppose any quantity we please to be the independant variable, since the substituted expressions represent the successive derivatives of y, whatever be the independant variable. If the expression involve dx, dy, d^2y, d^3y, &c. but not in the precise forms

$$\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \frac{d^3y}{dx^3}, \text{ \&c.}$$

then we substitute in this manner

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx}; \quad \therefore\ d^2y = dx\,d\left(\frac{dy}{dx}\right),$$

which we put for d^2y,

$$\frac{d^3y}{dx^3} = \frac{d\left\{\frac{d\left(\frac{dy}{dx}\right)}{dx}\right\}}{dx}; \quad \therefore\ d^3y = dx^2\,d\left\{\frac{d\left(\frac{dy}{dx}\right)}{dx}\right\},$$

which put for d^3y, and so on.

In this manner we may change the independant variable in any expression. The following examples will make this clear.

102. In the expression **Example.**

$$\frac{d^2y}{dx^2} - \frac{x}{1-x^2}\frac{dy}{dx} + \frac{y}{1-x^2} = u,$$

where x is the independant variable, to make θ the independant variable, $\cos\theta$ being equal to x.

Making the necessary substitutions for $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, we have

$$u = \frac{d\left(\frac{dy}{dx}\right)}{dx} - \frac{x}{1-x^2}\frac{dy}{dx} + \frac{y}{1-x^2}.$$

In this expression we are at liberty to assume any quantity we please to be the independant variable; let us suppose therefore that θ is, and then we have

$$dx = -\sin\theta d\theta, \quad \frac{dy}{dx} = -\frac{dy}{d\theta\sin\theta};$$

$$\therefore\ d\left(\frac{dy}{dx}\right) = -\frac{d^2y\sin\theta - dy\cos\theta d\theta}{d\theta\sin^2\theta},$$

since $d\theta$ is now constant.

$$\text{Hence}\quad \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d^2y\sin\theta - dy\cos\theta\, d\theta}{\ast\; d\theta^2\sin^3\theta} \qquad \ast\; d\theta^2 \sin^3\theta$$

$$-\frac{x}{1-x^2}\,\frac{dy}{dx} = \frac{\cos\theta}{\sin^2\theta}\,\frac{dy}{d\theta\sin\theta},$$

$$\frac{y}{1-x^2} = \frac{y}{\sin^2\theta}.$$

$$\text{Hence}\quad u = \left(\frac{d^2y}{d\theta^2} + y\right)\frac{1}{\sin^2\theta},$$

in which expression θ is supposed to be the independant variable.

Another Example. 103. Given the expression

$$\left(3\frac{d^2y}{dy^2} - 1\right)\frac{d^2y}{dy^2} - \frac{d^3y}{dy^3} = u,$$

in which x is the independant variable; to find what it becomes when y is the independant variable.

$$\text{Here put}\quad \frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx},\ \text{and}\ \therefore\ d^2y = dx\, d\left(\frac{dy}{dx}\right);$$

$$\text{also put}\quad \frac{d^3y}{dx^3} = \frac{d\left\{\frac{d\left(\frac{dy}{dx}\right)}{dx}\right\}}{dx},$$

$$\text{and}\ \therefore\ d^3y = dx^2 d\left\{\frac{d\left(\frac{dy}{dx}\right)}{dx}\right\}.$$

and then we have

$$u = \left\{3\frac{dx}{dy^2}d\left(\frac{dy}{dx}\right) - 1\right\}\frac{dx}{dy^2}d\left(\frac{dy}{dx}\right) - \frac{dx^2}{dy^3}d\left\{\frac{d\left(\frac{dy}{dx}\right)}{dx}\right\}.$$

In this expression we are at liberty to assume any quantity we please to be the independant variable, let us therefore suppose that y is, and then we have

$$d\left(\frac{dy}{dx}\right) = -\frac{dy\,d^2x}{dx^2}, \text{ since } dy \text{ is now constant;}$$

$$d\left\{\frac{d\left(\frac{dy}{dx}\right)}{dx}\right\} = -dy\left(\frac{d^3x}{dx^3} - \frac{3\,d^2x\,d^2x}{dx^4}\right).$$

Hence $u = \left(3\frac{d^2x}{dx\,dy} + 1\right)\frac{d^2x}{dx\,dy} + \frac{d^3x}{dx\,dy^2} - 3\frac{(d^2x)^2}{dx^2dy^2}$

$$= \frac{d^2x}{dx\,dy} + \frac{d^3x}{dx\,dy^2} = \frac{dy}{dx}\left(\frac{d^2x}{dy^2} + \frac{d^3x}{dy^3}\right),$$

in which expression y is the independant variable.

(For other examples see Appendix E).

The following articles have reference to the successive differentiation of functions of several variables.

†† 104. Suppose that y is some function of u and v, u and v being functions of x; then by Rule XV. Successive differentiation of $f(uv)$, u v being functions of x.

$$dy = d_u y\,du + d_v y\,dv;$$

therefore differentiating again

$$d^2y = d(d_u y\,du) + d(d_v y\,dv).$$

Now $d(d_u y\,du) = d(d_u y).du + d_u y.d^2u$

$= \{d_u(d_u y).du + d_v(d_u y).dv\}\,du + d_u y\,d^2u$, by Rule XV.

hence assuming $d_u^2 y$ to denote $d_u d_u y$, we have

$$d(d_u y\,du) = d_u^2y.du^2 + d_v d_u y.dudv + d_u y d^2u \;\ldots\; (1);$$

and similarly we have

$$d(d_v y\,dv) = d_v^2y.dv^2 + d_u d_v y.dudv + d_v y d^2v \;\ldots\; (2);$$

and adding (1) and (2), we obtain an expression for d^2y. This expression becomes somewhat simplified by means of the following article.

Proposition $d_u d_v y = d_v d_u y$.

†† 105. *To shew that* $d_u d_v y = d_v d_u y$.

Let $y = \phi(uv)$; then, by the definition of a partial derivative, if we put

$$\frac{\phi(u'v) - \phi(uv)}{u' - u} \sim d_u y = \psi(u'v),$$

$\psi(u'v)$ may be diminished *ad libitum* by sufficiently diminishing $u' - u$, without altering v; and this is true for all values of v: hence we may evidently diminish $\dfrac{\psi(u'v') - \psi(u'v)}{v' - v}$ *ad libitum*, by sufficiently diminishing $u' - u$ without altering v' or v; and this is true for all values of v' and v (except of course v' actually $= v$). Hence, since we may make $\dfrac{\psi(u'v') - \psi(uv)}{v' - v}$ differ as little as we please from $d_v\psi(u'v)$ by taking $v' - v$ small enough, it clearly must be possible to diminish $d_v\psi(u'v)$ *ad libitum*, by sufficiently diminishing $u' - u$; but

$$d_v\psi(u'v) = \frac{d_v\phi(u'v) - d_v\phi(uv)}{u' - u} \sim d_v d_u y;$$

hence, by Lemma VI, Cor. 2, $d_v d_u y$ must be the limiting value of $\dfrac{d_v\phi(u'v) - d_v\phi(uv)}{u' - u}$ when u' approaches u; but, by the definition of a partial derivative, this limiting value is represented by $d_u\{d_v\phi(uv)\}$, or $d_u d_v y$. Hence we have

$$d_u d_v y = d_v d_u y.$$

It appears therefore that whenever we have $d_u d_v$ written before any expression, we may write $d_v d_u$ instead of it, and *vice versâ:* i.e. the order in which we perform partial differentiations is indifferent.

Cor. $d_u^m d_v^n y = d_v^n d_u^m y$.

106. Differentiating the result $d_u d_v y = d_v d_u y$ with respect to u, we have

$$d_u^2 d_v y = d_u d_v d_u y = d_v d_u^2 y, \quad \text{putting } d_v d_u \text{ for } d_u d_v;$$

and differentiating this result, $d_u^3 d_v y = d_u d_v d_u^2 y = d_v d_u^3 y$ similarly, and so on, and in general

$$d_u^m d_v y = d_v d_u^m y.$$

It appears therefore that whenever we have $d_u^m d_v$ written before any expression, we may write $d_v d_u^m$ instead of it, and *vice versâ*.

Differentiating the result $d_v d_u^m y = d_u^m d_v y$ with respect to v, we have

$d_v^2 d_u^m y = d_v d_u^m d_v y = d_u^m d_v^2 y$, putting $d_u^m d_v$ instead of $d_v d_u^m$,

and again $d_v^3 d_u^m y = d_v d_u^m d_v^2 y = d_u^m d_v^3 y$ similarly,

and so on, and in general

$$d_v^n d_u^m y = d_u^m d_v^n y.$$

It appears therefore that whenever we have $d_v^n d_u^m$ written before any expression, we may write $d_u^m d_v^n$ instead of it.

(Another proof of this result will be given when we come to speak of series in Chapter IX.)

This result shews that when we successively differentiate an expression with respect to different variables, it is no matter in what order these differentiations are performed.

†† 107. We shall now return to (104). Successive differentiation of $f(uv)$.

d^2y is found by adding (1) and (2); hence, from what has been just proved, we have

$$d^2y = d_u^2y\,.\,du^2 + 2\,d_ud_vy.dudv + d_v^2y.dv^2 + d_uy.d^2u + d_vy.d^2v.$$

If we differentiate this result again in a similar manner, we obtain

$$d^3y = d_u^3y.du^3 + 3\,d_u^2d_vy.du^2dv + 3\,d_u^2d_vy.dudv^2 + d_v^3y.dv^3,$$

$$3\,d_u^2y\,.\,dud^2u + 3\,d_ud_vy\,.\,(d^2udv + dud^2v) + 3\,d_v^2y\,.\,dvd^2v$$

$$+\ d_uy\,.\,d^3u + d_vy\,.\,dv^3\,;$$

and so we may find d^4y, &c.; but the results become extremely complicated, and we shall not put them down.

Let $y = u^2 + v^2 + uv$, Example.

then $d_uy = 2u + v$, $d_u^2y = 2$, $d_vy = 2v + u$, $d_v^2y = 2$,

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$d_u d_v y = 1$; and therefore we have

$$d^2y = 2du^2 + 2du\,dv + 2dv^2 + (2u+v)\,d^2u + (2v+u)\,d^2v.$$

Again $d_u^3 y = 0$, $d_v^3 y = 0$, $d_u^2 d_v y = 0$, $d_u d_v^2 y = 0$;

and therefore we have

$$d^3y = 3(2du\,d^2u + d^2u\,dv + dv\,d^2u + 2dv\,d^2v) + (2u+v)\,d^3u + (2v+u)\,d^3v,$$

and so we may find d^4y, &c.

The method here given of finding the successive differentials of a function of several variables is useful only for general purposes. In particular cases we may always find the successive differentials more readily by the common rules of differentiation. (See Appendix F where examples of this are given).

Successive differentiation of an equation between y and x.

†† 108. If U be a function of x and y, and we put $U = 0$, then we have by Rule XVI.

$$d_x U \,.\, dx + d_y U \,.\, dy = 0,$$

and differentiating as in (107) we have, supposing dx constant,

$$d_x^2 U \,.\, dx^2 + 2d_x d_y U \,.\, dx\,dy + d_y^2 U \,.\, dy^2 + d_y U \,.\, d^2y = 0,$$

and so we may go on, and by this means find $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \dfrac{d^3y}{dx^3}$ ... &c. from the equation $U = 0$.

Ex. Let $x^3 - 3axy + y^3 = U = 0$,

then $d_x U = 3x^2 - 3ay$, $d_x^2 U = 6x$, $d_y U = 3y^2 - 3ax$,

$d_y^2 U = 6y$, $d_x d_y U = -3a$: we have therefore

$$(x^2 - ay)\,dx + (y^2 - ax)\,dy = 0 \text{ .... } (1)$$

$$2x\,dx^2 - 2a\,dx\,dy + 2y\,dy^2 + (y^2 - ax)\,d^2y = 0 \ldots (2),$$

from (1) we get $\dfrac{dy}{dx} = -\dfrac{x^2 - ay}{y^2 - ax}$,

and then from (2)

$$2x + 2a\,\frac{x^2 - ay}{y^2 - ax} + 2y\,\frac{(x^2 - ay)^2}{(y^2 - ax)^2} + (y^2 - ax)\,\frac{d^2y}{dx^2} = 0,$$

which gives $\frac{d^2y}{dx^2}$; and so we may find $\frac{d^3y}{dx^3}$, &c.

(See Appendix F.)

†† 109. If y be a function of u and v, u and v being independant variables, then supposing du and dv constant (see 94), we have, as in (107) Successive differentiation of a function of two independant variables.

$$dy = d_u y\,.\,du + d_v y\,.\,dv,$$

$$d^2y = d_u^2 y\,.\,du^2 + 2d_u d_v y\,.\,du\,dv + d_v^2 y\,.\,dv^2,$$

$$d^3y = d_u^3 y\,.\,du^3 + 3d_u^2 d_v y\,.\,du^2\,dv + 3d_u d_v^2 y\,.\,du\,dv^2 + d_v^3 y\,.\,dv^3,$$

&c. .... &c. ....

We perceive that the coefficients of the terms in the second members of these equations are respectively those in the expansions of $1 + x$, $(1 + x^2)$, $(1 + x)^3 \ldots$

Let 1, A, B, $C \ldots$ be the coefficients in the expansion of $(1 + x)^n$, and let us assume that

$$d^n y = d_u^n y\,.\,du^n + A\,d_u^{n-1} d_v y\,.\,du^{n-1}dv + B\,d_u^{n-2} d_v^2 y\,.\,du^{n-2}dv^2 + \ldots$$

$$\text{then } d^{n+1}y = d_u(d^n y)\,.\,du + d_v(d^n y)\,.\,dv$$

$$= d_u^{n+1} y\,.\,du^{n+1} + \left.\begin{matrix}A\\+1\end{matrix}\right| d_u^n d_v y\,.\,du^n dv + \left.\begin{matrix}B\\+A\end{matrix}\right| d_u^{n-1} d_v^2 y\,.\,du^{n-1}dv^2 + \ldots$$

Now 1, $A+1$, $B+A$, &c. we know to be the coefficients in $(1 + x)^{n+1}$ expanded: hence it appears that if the law we have assumed be true for n it is true also for $n + 1$; but we know it to be true for 1, 2, 3; therefore it is true in general. We have therefore

$$d^n y = d_u^n y\,.\,du^n + \frac{n}{1}\,d_u^{n-1} d_v y\,.\,du^{n-1}dv + \frac{n(n-1)}{1\,.\,2}\,.\,d_u^{n-2} d_v^2 y\,.\,du^{n-2}dv^2 + \ldots$$

Symbolical expression for $d^n y$.

††110. This result may be briefly expressed thus,

$$d^n y = (d_u \,.\, du + d_v \,.\, dv)^n . y.$$

By which formula, we do not mean that $d_u \,.\, du$ and $d_v \,.\, dv$ are actually quantities whose sum is raised to the n^{th} power and multiplied into y; but simply this, that if the symbols $d_u \,.\, du$ and $d_v \,.\, dv$ be connected together by the sign $+$, and raised to the n^{th} power in the same manner as if they were ordinary quantities, and if y be written after each term of the result; then the expression so obtained is the proper expression for $d^n y$.

Expression for $d^n y$, when y is a function of more than two independant variables.

††111. We may prove in the same manner that if y be a function of several independant variables u, v, w...&c., then

$$d^n y = (d_u \,.\, du + d_v \,.\, dv + d_w \,.\, dw \ldots)^n . y.$$

CHAPTER VIII.

CERTAIN LEMMAS UPON WHICH THE APPLICATION OF THE DIFFERENTIAL CALCULUS IN MANY CASES DEPENDS.

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112. HAVING now concluded what may be considered the first part of the differential calculus, namely that in which we determine a set of rules whereby differentiation may be performed in all cases with facility; we proceed to the second part, in which we shall shew some of the principal uses which may be made of the differential calculus in pure mathematics. But we must previously prove the following lemmas upon which the application of the differential calculus in a great measure depends.

113. *If* f(x) *changes its sign when* x *passes* through the value* a, *then* f(a) *must be* 0 *or* ∞. Lemma XVII.

For $f(a)$ is the limiting value of $f(x)$ when x approaches a (Lemma III), and therefore by Lemma IV $f(x)$ has the same sign as $f(a)$ for all values of x taken sufficiently near a; hence, if $f(a)$ be positive, $f(x)$ is positive for all values of x taken sufficiently near a, and therefore cannot change its sign when x passes through the value a; and the same is true if $f(a)$ be negative. If therefore $f(x)$ changes its sign when x passes through the value a, $f(a)$ can neither be positive nor negative; i.e. it must be 0 or $\frac{1}{0}$. Q. E. D.

114. It is important to observe, that what we have proved here is, not that $f(x)$ *must* change its sign when $f(a)$ is 0 or ∞, but that if *it does change* its sign, $f(a)$ *must* be 0 or ∞. $f(x)$ does not necessarily change its sign when $f(a)$ is 0 or ∞.

* By saying that, "$f(x)$ changes its sign when x passes through the value a," we mean that when we give x any value a little greater than a, $f(x)$ has a different sign to what it has when we give x any value a little less than a.

Thus, if $f(x) = (x - a)^3$, $f(x)$ changes its sign when x passes through the value a, and here $f(a) = 0$. If $f(x) = \frac{1}{(x-a)^3}$, $f(x)$ changes its sign when x passes through a, and here $f(a) = \infty$. If $f(x) = (x - a)^2$, $f(a) = 0$, but $f(x)$ does not change its sign; and if $f(x) = \frac{1}{(x-a)^2}$, $f(a) = \infty$, but $f(x)$ does not change its sign.

Lemma XVIII.

115. *If we suppose* x *to increase continually,* f(x) *is increasing as long as* f′(x) *continues positive, and is diminishing as long as* f′(x) *continues negative.*

For, by Lemma IV, $\frac{f(x') - f(x)}{x' - x}$ has the same sign as $f'(x)$, for all values of x' taken sufficiently near x: therefore if $f'(x)$ be positive, $f(x') - f(x)$ has the same sign as $x' - x$, and therefore if x' be $> x$, $f(x')$ is $> f(x)$; i.e. $f(x)$ increases when x increases. And similarly, if $f'(x)$ be negative, $f(x') - f(x)$ has the opposite sign to that of $x' - x$, and therefore $f(x')$ is $< f(x)$ if x' be $> x$; i.e. $f(x)$ diminishes when x increases. Hence the truth of the lemma is manifest.

$f'(x)$ measures the rate of variation of $f(x)$ as compared with x.

116. Since $\frac{f(x') - f(x)}{x' - x}$ is very nearly the same thing as $f'(x)$ for all values of x' taken sufficiently near x, it is evident that $f'(x)$ is very nearly the ratio of any small change in $f(x)$ to the corresponding change in x; and therefore the greater $f'(x)$ is, the greater will be the *rate*, so to speak, at which $f(x)$ varies as compared with x.

Hence $f'(x)$, by its sign shews whether $f(x)$ is increasing or diminishing, and by its magnitude the rate at which that increase or diminution takes place.

If we suppose x diminish continually, it is evident that $f(x)$ diminishes or increases according as $f'(x)$ is positive or negative.

117. *If* f (x) *and* ϕ(x) *be any two functions of* x, *then* Lemma XIX.
$\frac{f'(x)}{\phi'(x)}$ *is in general the limiting value of* $\frac{f(x') - f(x)}{\phi(x') - \phi(x)}$
when x′ *approaches* x.

For $$\frac{f(x') - f(x)}{\phi(x') - \phi(x)} = \frac{\dfrac{f(x') - f(x)}{x' - x}}{\dfrac{\phi(x') - \phi(x)}{x' - x}}$$

and the limiting value of the second member, by Lemma VIII, is evidently $\frac{f'(x)}{\phi'(x)}$ in general. Q. E. D.

We say "*in general,*" because $\frac{f'(x)}{\phi'(x)}$ may become illusory for some particular value of x, and then of course this Lemma fails. In such a case the following Lemma will take its place.

118. *If* a *be any value of* x *which makes* $\frac{f'(x)}{\phi'(x)}$ Lemma XX.
illusory, then the limiting value of $\frac{f'(x)}{\phi'(x)}$ *when* x *approaches* a *is also the limiting value of* $\frac{f(x) - f(a)}{\phi(x) - \phi(a)}$ *when* x *approaches* a.

By the last proposition $\frac{f'(x)}{\phi'(x)}$ is in general the limiting value $\frac{f(x') - f(x)}{\phi(x') - \phi(x)}$ when x' approaches x; therefore

$$\frac{f(x') - f(x)}{\phi(x') - \phi(x)} \sim \frac{f'(x)}{\phi'(x)}$$

may be diminished *ad libitum* by sufficiently diminishing $x' \sim x$; and this is true for all values of x which do not make $\frac{f'(x)}{\phi'(x)}$ illusory; therefore it is true if we put $x' = a$ and sufficiently diminish $x \sim a$; i. e. we may diminish

$$\frac{f(a)-f(x)}{\phi(a)-\phi(x)} \sim \frac{f'(x)}{\phi'(x)}$$

ad libitum, by sufficiently diminishing $x \sim a$; hence by Lemma VI, Cor. 2, the limiting value of $\frac{f'(x)}{\phi'(x)}$ is also the limiting value of $\frac{f(a)-f(x)}{\phi(a)-\phi(x)}$ or $\frac{f(x)-f(a)}{\phi(x)-\phi(a)}$ when x approaches a. Q. E. D.

Cor. 1. 119. *If* a *be a value of* x *which makes the quantities* $f^1(a)$, $f^2(a) \ldots f^{n-1}(a)$, $\phi^1(a)$, $\phi^2(a) \ldots \phi^{n-1}(a)$, *each zero, and if* $\frac{f^n(a)}{\phi^n(a)}$ *be not illusory; then* $\frac{f^n(a)}{\phi^n(a)}$ *is the limiting value of* $\frac{f(x)-f(a)}{\phi(x)-\phi(a)}$ *when* x *approaches* a.

For, by the Lemma, when x approaches a,

$$\text{lim. val. of } \frac{f(x)-f(a)}{\phi(x)-\phi(a)} = \text{lim. val. of } \frac{f'(x)}{\phi'(x)},$$

$$\text{or of } \frac{f'(x)-f'(a)}{\phi'(x)-\phi'(a)},$$

since $f'(a)$ and $\phi'(a)$ are each zero,

$$= \text{lim. val. of } \frac{f^2(x)}{\phi^2(x)} \text{ by the Lemma,}$$

$$\text{or of } \frac{f^2(x)-f^2(a)}{\phi^2(x)-\phi^2(a)},$$

since $f^2(a)$ and $\phi^2(a)$ are each zero;

and so on, till we come to

$$\text{lim. val. of } \frac{f^{n-1}(x)-f^{n-1}(a)}{\phi^{n-1}(x)-\phi^{n-1}(a)},$$

$$\text{which } = \frac{f^n(a)}{\phi^n(a)}, \text{ by Lemma XIX.}$$

Hence $\dfrac{f^n(a)}{\phi^n(a)}$ is the limiting value of $\dfrac{f(x)-f(a)}{\phi(x)-\phi(a)}$ when x approaches a. Q. E. D.

120. *If* a *be a value of* x *which makes* $f^1(a)$, $f^2(a)$ Cor. 2.
$\ldots f^{n-1}(a)$ *each zero, then* $\dfrac{f^n(a)}{\Gamma n}$ *is the limiting value of* $\dfrac{f(x)-f(a)}{(x-a)^n}$ *when* x *approaches* a.

This is easily proved by putting $\phi(x)=(x-a)^n$ in the last Cor., and therefore $\phi(x)-\phi(a)=(x-a)^n$, $\phi^1(a)=0$, $\phi^2(a)=0\ldots\phi^{n-1}(a)=0$, $\phi^n(a)=\Gamma n$; which evidently gives us $\dfrac{f^n(a)}{\Gamma n}$ for the limiting value of $\dfrac{f(x)-f(a)}{(x-a)^n}$ when x approaches a. Q. E. D.

Conversely: *If a finite quantity* A *be the limiting value* Cor. 3.
of $\dfrac{f(x)-f(a)}{(x-a)^n}$ *when* x *approaches* a, *then must* $f^1(a)=0$, $f^2(a)=0$, $f^3(a)=0\ldots f^{n-1}(a)=0$, *and* $f^n(a)=\Gamma n.A$.

For if we have $f^1(a)=0$, $f^2(a)=0\ldots f^{r-1}(a)=0$, but $f^r(a)$ not $=0$, r being any integer less than n; then the limiting value of $\dfrac{f(x)-f(a)}{(x-a)^r}$ when x approaches a will be $\dfrac{f^r(a)}{\Gamma r}$, and therefore that of $\dfrac{f(x)-f(a)}{(x-a)^n}$, which

$$=\frac{1}{(x-a)^{n-r}}\cdot\frac{f(x)-f(a)}{(x-a)^r},$$

will be $\dfrac{1}{0}\cdot\dfrac{f^r(a)}{\Gamma r}$ (Lemma VIII.); which is infinite, contrary to hypothesis. Hence the truth of the Cor. is evident.

121. *If* $f^1(a)$, $f^2(a)\ldots f^{n-1}(a)$, *be each zero, and* Cor. 4.
$f^n(a)$ *not zero, then, for all values of* x *taken sufficiently near* a, $f(x)-f(a)$ *has the same sign as* $f^n(a)(x-a)^n$.

For, by Lemma IV, $\dfrac{f(x)-f(a)}{(x-a)^n}$ has the same sign as its limit $\dfrac{f^n(a)}{\Gamma n}$ for all values of x sufficiently near a, and therefore $f(x)-f(a)$ has the same sign as $f^n(a)(x-a)^n$. Q. E. D.

Lemma XXI.

122. *If* f(x) *be any function of* x *we may assume in general that*

$$f(x)=f(a)+f'(a)\frac{x-a}{\Gamma 1}+f^2(a)\frac{(x-a)^2}{\Gamma 2}+f^3(a)\frac{(x-a)^3}{\Gamma 3}\ \ldots\ldots$$
$$+\left\{f^n(a)+Q\right\}\frac{(x-a)^n}{\Gamma n},$$

where Q *is some function of* x *which may be diminished* ad libitum *by sufficiently diminishing* x ~ a.

For assume $F(x)$ to represent the quantity

$$f(x)-\left\{f(a)+f'(a)\frac{x-a}{\Gamma 1}+f^2(a)\frac{(x-a)^2}{\Gamma 2}\ldots f^n(a)\frac{(x-a)^n}{\Gamma n}\right\}\ldots(1),$$

then, by differentiating this expression successively and putting $x=a$, it is easy to see that

$$F(a)=0,\quad F'(a)=0,\quad F^2(a)=0\ \ldots\ldots\ F^n(a)=0.$$

Hence, by Lemma XX, Cor. 2, 0 must be the limiting value of $\dfrac{F(x)}{(x-a)^n}$ when x approaches a, and therefore $\dfrac{F(x)}{(x-a)^n}$ may be diminished *ad libitum* by sufficiently diminishing $x \sim a$. If therefore we put

$$\frac{F(x)}{(x-a)^n}=\frac{Q}{\Gamma n},\ \text{ or } F(x)=Q\,.\,\frac{(x-a)^n}{\Gamma n},$$

and substitute for $F(x)$ its value (1) we have

$$f(x)-\left\{f(a)+f'(a)\frac{x-a}{\Gamma 1}+f^2(a)\frac{(x-a)^2}{\Gamma 2}\ldots\ldots\right.$$
$$\left.f^n(a)\frac{(x-a)^n}{\Gamma n}\right\}=Q\,\frac{(x-a)^n}{\Gamma n},$$

$$\text{or } f(x) = f(a) + f'(a)\frac{x-a}{\Gamma 1} + f^2(a)\frac{(x-a)^2}{\Gamma 2} \dots\dots$$

$$+ \{f^n(a) + Q\}\frac{(x-a)^n}{\Gamma n},$$

where Q, since it $= \Gamma n \dfrac{F(x)}{(x-a)^n}$, is some function of x which may be diminished *ad libitum* by sufficiently diminishing $x \sim a$. Q. E. D.

This reasoning fails when any of the quantities $f(a)$, $f'(a) \dots f^n(a)$ are infinite; for then we cannot assert that all the quantities $F(a)$, $F'(a)$, $F''(a) \dots F^n(a)$ are zero; which is essential to the proof.

†† 123. *If* M *be the least value of* $f^n(x) - f^n(a)$, *and* N *the greatest, for all values of* x *between* a *and another value* b; *then* Q *lies between* M *and* N, *for all values of* x *between* a *and* b. Lemma XXII.

Let C be any constant, and let us write down the expression $F(x) - C\dfrac{(x-a)^n}{\Gamma n}$ and its successive differential coefficients as follows:

$$F(x) - C\frac{(x-a)^n}{\Gamma n} \dots\dots (n),$$

$$F'(x) = C\frac{(x-a)^{n-1}}{\Gamma(n-1)} \dots\dots (n-1),$$

$$F^2(x) - C\frac{(x-a)^{n-2}}{\Gamma(n-2)} \dots\dots (n-2),$$

$$\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots$$

$$F^{n-1}(x) - C(x-a) \dots\dots\dots\dots (1),$$

$$F^n(x) - C \quad \dots\dots\dots\dots\dots\dots\dots (2).$$

Now $F^n(x) = f^n(x) - f^n(a)$ evidently; hence putting $C = M$, which is the least value of $f^n(x) - f^n(a)$ and there-

fore of $F^n(x)$ for all values of x between a and b, it is clear that if we suppose x to increase from a to b, (supposing b greater than a,) the expression (0) is always positive, and therefore the expression (1) is always increasing by Lemma XVIII, but (1) is zero when $x = a$, therefore (1) is always positive; and therefore in the same way we may shew that (2) is always positive; and so on, and finally that (n) is always positive, and therefore that $\Gamma n \dfrac{F^n(x)}{(x-a)^n}$, or Q, is always greater than C, i.e. M.

In exactly the same way if we put $C = N$, we may shew that the expressions (0), (1), (2) ... (n) are all negative while x increases from a to b, and that therefore Q is always less than N.

If b be less than a we may shew in exactly the same way that when x decreases from a to b the expressions (0), (1), (2) ..., and finally (n) are all negative if $C = M$, and positive when $C = N$; and therefore that Q always lies between M and N.

Hence it appears that Q lies between M and N for all values of x between a and b. Q. E. D.

Lemma XXIII. 124. *When a function* f(x) *becomes infinite for a particular value of* x, *all its differential coefficients must also become infinite.*

For $\dfrac{f(x)-f(a)}{x-a}$ may be made to differ as little as we please from $f'(a)$ by making x approach a; hence if $f(a)$ and therefore $\dfrac{f(x)-f(a)}{x-a}$ be infinite, and at the same time $f'(a)$ finite, we may make infinity differ as little as we please from a finite quantity; which is absurd. Hence $f'(a) = \infty$: and therefore by similar reasoning $f^2(a) = \infty$ and $f^3(a) = \infty$, and so on.

This reasoning does not hold if $a = \infty$.

CHAPTER IX.

APPLICATION OF THE DIFFERENTIAL CALCULUS TO THE DEVELOPEMENT OF FUNCTIONS; WITH SOME PRELIMINARY REMARKS RESPECTING SERIES.

WE now proceed to shew some of the principal uses which may be made of the differential calculus in pure mathematics, and in the first place its application to the developement of functions in series.

125. It will be necessary to make a few preliminary remarks on the nature of an ordinary series of the form Preliminary remarks respecting series.

$$A_0 + A_1x + A_2x^2 \ldots + A_nx^n + \&c. \ldots;$$

and to settle precisely what we mean by the equation

$$f(x) = A_0 + A_1x + A_2x^2 \ldots + A_nx^n + \&c. \ldots (1)$$

which expresses the developement of a function $f(x)$ in a series of powers of x.

126. In the first place we may remark that the sign = here, as well as elsewhere, always signifies actual equality; for mathematicians always consider themselves at liberty, when they have two expressions of any kind connected by the sign =, to use them indifferently for each other, and to substitute one for the other in any calculation; which certainly they have no right to do, if the sign = does not always signify actual equality. We shall therefore always use = as the sign of actual equality. = always denotes actual equality.

127. Secondly, by the second member of the equation (1) we do not *in general* mean a series of terms infinite in number, but a series of terms carried on according to a certain law to any number we please, with a term at the end of a dif- We must not *in general* suppose a series to consist of an infinite number of

terms, but of a finite number with a remainder. ferent kind from the rest, which is commonly called a remainder: the former terms are represented by

$$A_0 + A_1 x + A_2 x \ldots + A_n x^n,$$

n being any integer, and the remainder is represented by the "+ &c.," which is written after the term $A_n x^n$. The remainder we shall often denote by writing $+ R$ instead of "+ &c."

Thus when we say that

$$\frac{1}{1-x} = 1 + x + x^2 \ldots + x^n + \text{\&c.} \ldots\ldots (2)$$

we do not *in general* mean to assert that $1 + x + x^2 \ldots$ to an infinite number of terms $= \dfrac{1}{1-x}$, for if so, suppose $x = 2$ and then we have $1 + 2 + 8 + 16 \ldots$ to an infinite number of terms $= -1$ which is manifestly absurd. But we simply mean to assert that

$$\frac{1}{1-x} = 1 + x + x^2 \ldots + x^n + R \ldots\ldots (3)$$

where n is a number as large as we please, and R a certain quantity which must be always added to the series to make the second member of the equation equal to the first. In fact the "+ &c." in equation (2), and the $+ R$ in equation (3) mean the same thing.

In the present instance the value of R is $\dfrac{x^{n+1}}{1-x}$, for it is easy to see that

$$\frac{1}{1-x} = 1 + x + x^2 \ldots x^n + \frac{x^{n+1}}{1-x},$$

and thus when $x = 2$ we have

$$-1 = 1 + 2 + 8 + 16 \ldots 2^n - 2^{n+1},$$

in which equation there is no such absurdity as there is when the number of terms is considered infinite.

We may under cer- 128. Thirdly, if it should so happen that R is a quantity

which may be diminished *ad libitum* by sufficiently increasing n, then we may assert that

tain circumstances assert that a series consists of an infinite number of terms.

$$f(x) = A_0 + A_1x + A_2x^2 \ldots$$

to an infinite number of terms: or perhaps more correctly, that $f(x)$ is the limiting value of $A_0 + A_1x + A_2x^2 \ldots A_nx^n$ when n approaches infinity.

For $f(x) - (A_0 + A_1x + A_2x^2 \ldots A_nx^n)$, since it $= R$, may be diminished *ad libitum* by sufficiently increasing n; and therefore, by Lemma VI, Cor. 2, $f(x)$ is the limiting value of $A_0 + A_1x + A_2x^2 \ldots A_nx^n$ when n approaches ∞. When we assert therefore that

$$f(x) = A_0 + A_1x + A_2x^2 \ldots \textit{ad infinitum},$$

we simply mean that $f(x)$ is the limiting value of

$$A_0 + A_1x + A_2x^2 + \ldots A_nx^n,$$

when n approaches ∞.

Definition of a converging and of a diverging series.

129. When R may be diminished *ad libitum* by sufficiently increasing n, the series

$$A_0 + A_1x + A_2x^2 \ldots A_nx^n + R$$

is said to be a *converging series:* otherwise it is called a *diverging series.*

Hence we may assert that

$$f(x) = A_0 + A_1x + A_2x^2 \ldots \textit{ad infinitum},$$

when the series is converging, but not otherwise.

A function may be developed in an infinite number of different series.

130. In the fourth place we may remark, that we may always assume

$$f(x) = A_0 + A_1x + A_2x^2 \ldots A_nx^n + R,$$

where A_0, A_1, $A_2 \ldots A_n$ are any constants whatever. For we may always give R such a value as will make the second member of this equation to coincide with the first, no matter what values we assign to A_0, A_1, $A_2 \ldots A_n$.

Thus we may assume

$$\frac{1}{1-x} = 1 + x + x^2 + 5x^3 - 7x^4 + R,$$

for by giving R the value $-\frac{4x^3-12x^4+7x^5}{1-x}$ the second member of this equation becomes identical with the first.

We may therefore expand any function $f(x)$ in an infinite number of different ways in a series of the form

$$A_0 + A_1x + A_2x^2 \ldots A_nx^n + R \ldots (1).$$

Distinction between a perfect and an imperfect developement.

131. And here, in the fifth place, we may make an important distinction; namely, the distinction between what we may call a *perfect* and an *imperfect developement*.

The series $A_0 + A_1x + A_2x^2 \ldots A_nx^n + R$ we shall call a perfect developement of $f(x)$, when R is of such a nature, that zero is the limiting value of $\frac{R}{x^n}$ when x approaches zero; and if this limiting value be not zero we shall call the series an imperfect developement.

That we are justified in making this distinction, and regarding it as a very important distinction, is evident from the following theorem.

The principle of indeterminate coefficients holds for a perfect but not for an imperfect developement.

132. *The principle of indeterminate coefficients holds for a perfect developement, but not for an imperfect.*

By the principle of indeterminate coefficients we mean this: that if we have a series

$$A_0 + A_1x + A_2x^2 \ldots + A_nx^n + R,$$

which is proved to be zero for all values of x, then must

$$A_0 = 0, \quad A_1 = 0, \quad A_2 = 0 \;\ldots\; A_n = 0.$$

Let $A_0 + A_1x + A_2x^2 \ldots A_nx^n + R$ be a perfect series, which is proved to be zero for all values of x, then must

$$\frac{R}{x^n} = -\frac{A_0 + A_1x + A_2x^2 \ldots A_nx^n}{x^n},$$

for all values of x, except of course the value zero: but the series being perfect, the limiting value of $\frac{R}{x^n}$, when x approaches 0 is zero; hence, by Lemma VI, Cor. 1, zero must be the limiting value of

$$\frac{A_0 + A_1 x + A_2 x^2 \dots A_n x^n}{x^n}$$

when x approaches 0: now if A_0 be not zero,

$$\frac{A_0 + A_1 x \dots A_n x^n}{x^n} \text{ becomes } \frac{A_0}{0} \text{ or } \infty,$$

when $x = 0$, and its limiting value cannot be zero when x approaches zero; therefore $A_0 = 0$, and we have

$$\frac{A_0 + A_1 x \dots A_n x^n}{x^n} = \frac{A_1 + A_2 x \dots A_n x^{n-1}}{x^{n-1}},$$

except of course when x actually $= 0$.

Hence zero must be the limiting value of this latter quantity when x approaches 0, which as before cannot be true unless $A_1 = 0$: and so we may go on and prove that $A_3 = 0$, $A_4 = 0 \dots$; and lastly, that

$$\frac{A_0 + A_1 x \dots A_n x^n}{x^n} = A_n,$$

the limiting value of which cannot be zero unless $A_n = 0$. Hence the principle of indeterminate coefficients holds for a perfect series.

But the principle of indeterminate coefficients does not hold for an imperfect developement.

For by what we have just proved, it is evident that if A_0, A_1, $A_2 \dots A_n$ be each zero, then the limiting value of $\frac{R}{x^n}$ when x approaches zero must be zero, and therefore the series must be perfect. If therefore the series be not perfect, A_0, $A_1 \dots A_n$ cannot be each zero, i. e. the principle of indeterminate coefficients does not hold.

Hence I think the importance of distinguishing series into perfect and imperfect is manifest.

A function which has only one value for each value of x can be developed in only perfect series.

133. Lastly, we may remark, that if $f(x)$ be a function which has only one value for each value of x, it cannot be developed in more than one perfect series.

For if possible let the two different perfect series

$$A_0 + A_1 x + A_2 x^2 \ldots A_n x^n + R, \text{ and } B_0 + B_1 x + B_2 x^2 \ldots B_n x^n + S,$$

be both equal to $f(x)$, then since $f(x)$ has only one value for each value of x, these series must be equal to each other; and therefore we have

$$A_0 - B_0 + (A_1 - B_1)x + (A_2 - B_2)x^2 \ldots + (A_n - B_n)x^n + R - S = 0.$$

Now when x approaches 0, the limiting values of $\frac{R}{x^n}$ and of $\frac{S}{x^n}$ are zero, therefore so also is that of $\frac{R-S}{x^n}$: hence, by what has been proved in the preceding Article, we must have $A_0 = B_0$, $A_1 = B_1$, $A_2 = B_2 \ldots A_n = B_n$, and therefore $R = S$; hence the two series are identical; and therefore $f(x)$ cannot be developed in two different perfect series.

Having made these preliminary remarks, we now proceed to shew the manner in which the Differential Calculus may be applied to the developement of functions in perfect series.

Proposition. To develope $f(a+h)$ in a perfect series of powers of h.

134. *To develope* f (a + h) *in a perfect series of powers of* h.

In Lemma XXI. put $a + h$ for x, and we have

$$f(a+h) = f(a) - f'(a)\frac{h}{\Gamma 1} + f^2(a)\frac{h^2}{\Gamma 2} \ldots f^n(a)\frac{h^n}{\Gamma n} + Q\frac{h^n}{\Gamma n} \quad (1),$$

here $Q\frac{h^n}{\Gamma n}$ is the remainder; call it R, then $\frac{R}{h^n} = \frac{Q}{\Gamma n}$; now zero is the limiting value of Q when x approaches a, i. e. when h approaches zero, (see Lemma XXI.), and therefore the same is true of $\frac{R}{h^n}$; hence (1) is a perfect developement. Since

Lemma XXI. fails when any of the quantities $f(a)$, $f'(a)$, $f^2(a) \dots f^n(a)$ become infinite, this developement fails under the same circumstances.

This developement is known by the name of Taylor's Series; when $a = 0$ it is called Maclaurin's Series, in which case it becomes Taylor's and Maclaurin's series.

$$f(h) = f(0) + f'(0)\frac{h}{\Gamma 1} + f^2(0)\frac{h^2}{\Gamma 2} \dots + f^n(0)\frac{h^n}{\Gamma n} + Q\frac{h^n}{\Gamma n}.$$

We may write the developement (1) thus:

$$f(a+h) = f(a) + f'(a)\frac{h}{\Gamma 1} + f^2(a)\frac{h^2}{\Gamma 2} \dots + f^n(a)\frac{h^n}{\Gamma n} + \&c.$$

remembering what "+ &c." means.

The following are very important examples of this developement. Examples.

135. Let $f(x) = x^m$; and therefore Developement of $(1+x)^m$.

$$f^n(x) = m(m-1)(\dots)(m-n+1)x^{m-n};$$

then putting 1 for a in the developement (1), we find

$$(1+h)^m = 1 + \frac{m}{1}h + \frac{m(m-1)}{\Gamma 2}h^2 \dots + \frac{m(m-1)(\dots.)(m-n+1)}{\Gamma n}h^n + \&c.$$

which is the binomial theorem for any value of the index.

Let $f(x) = \log x$, and $\therefore f^n(x) = \Gamma(n-1)(-x)^{-n}$; then putting 1 for a in the developement (1), we find Developement of $\log(1+x)$.

$$\log(1+h) = \frac{h}{1} - \frac{h^2}{2} + \frac{h^3}{3} \dots + \frac{(-h)^n}{n} + \&c.$$

Let $f(x) = c^x$, and $\therefore f^n(x) = k^n c^x$, where $k = \log c$, then putting $a = 0$ in the developement (1), we find Developement of c^x.

$$c^h = 1 + \frac{kh}{\Gamma 1} + \frac{k^2h^2}{\Gamma 2} \dots + \frac{k^n h^n}{\Gamma n} + \&c.$$

Developement of $\sin x$, $\cos x$, $\tan^{-1}x$.

Let $f(x) = \sin x$, and $\therefore\ f^n(x) = \sin\left(x + \frac{n\pi}{2}\right)$; then putting $a = 0$ in the developement (1), we find

$$\sin h = \frac{h}{\Gamma 1} - \frac{h^3}{\Gamma 3} + \frac{h^5}{\Gamma 5} \dots + (-)^n \frac{h^{2n+1}}{\Gamma(2n+1)} + \&c.$$

Similarly we find

$$\cos h = 1 - \frac{h^2}{\Gamma 2} + \frac{h^4}{\Gamma 4} \dots + (-)^n \frac{h^{2n}}{\Gamma(2n)} + \&c.$$

$$\tan^{-1} h = \frac{h}{1} - \frac{h^3}{3} + \frac{h^5}{5} \dots + (-)^n \frac{h^{2n+1}}{\Gamma(2n+1)} + \&c.$$

For more examples of the use of this developement, see Appendix G.

Proposition. Estimation of the error committed when the remainder in Taylor's series is neglected.

††136. *If we neglect the remainder* $Q\frac{h^n}{\Gamma n}$ *in the developement* (1) *last article, the error we commit in so doing lies between* $M\frac{h^n}{\Gamma n}$ *and* $N\frac{h^n}{\Gamma n}$, *where* M *and* N *are the greatest and least values of* $f^n(x) - f^n(a)$ *for all values of* x *between* a *and* a + h.

This follows immediately from Lemma XXII, if we put $b = a + h$: for it is there proved that Q lies between M and N for all values of x between a and b (i. e. $a + h$), and therefore for the particular value $a + h$; consequently the remainder $Q\frac{h^n}{\Gamma n}$ in the developement (1) must lie between $M\frac{h^n}{\Gamma n}$ and $N\frac{h^n}{\Gamma n}$.

Examples.

Ex. 1. Let $f(x) = \log x$, $a = 1$, $h = \frac{1}{10}$, $n = 5$; then

$$f^n(x) = (-)^{n-1}\frac{\Gamma(n-1)}{x^n}; \quad \therefore\ f^5(x) - f^5(1) = \Gamma 4 \cdot \left(\frac{1}{x^5} - 1\right):$$

the greatest and least values of this for all values of x between

1 and $1+\dfrac{1}{10}$ are evidently 0 and $\Gamma 4\left\{\dfrac{1}{\left(1+\frac{1}{10}\right)^5}-1\right\}$. Hence the error lies between 0 and $\dfrac{1}{5}\left(\dfrac{1}{11^5}-\dfrac{1}{10^5}\right)$; and we have therefore

$$\log\left(1+\frac{1}{10}\right)=\frac{1}{10}-\frac{1}{2}\frac{1}{10^2}+\frac{1}{3}\frac{1}{10^3}-\frac{1}{4}\frac{1}{10^4}+\frac{1}{5}\frac{1}{10^5}$$

$$+\text{ an error between 0 and }\frac{1}{5}\left(\frac{1}{11^5}-\frac{1}{10^5}\right).$$

Since $\dfrac{1}{11^5}=.000006\ldots$ this error lies between 0 and $-\dfrac{.000004\ldots}{5}$ which has no significant digit in the first 6 decimal places.

Ex. 2. Let $f(x)=x^{\frac{1}{2}}$, $a=1$, $h=\dfrac{1}{10}$, $n=5$; then

$$f^n(x)=\tfrac{1}{2}\left(\tfrac{1}{2}-1\right)(\ldots)\left(\tfrac{1}{2}-n+1\right)x^{\frac{1}{2}-n};$$

$$\therefore f^5(x)-f^5(1)=\frac{1.3.5.7}{2^5}\left(\frac{1}{x^{\frac{9}{2}}}-1\right):$$

the greatest and least values of this for all values of x between 1 and $1+\dfrac{1}{10}$ are 0 and

$$\frac{1.3.5.7}{2^5}\left\{\frac{1}{\left(1+\frac{1}{10}\right)^{\frac{9}{2}}}-1\right\}.$$

Hence the error lies between 0 and

$$\frac{1.3.5.7}{2^5\,\Gamma 5}\left\{\frac{1}{\left(1+\frac{1}{10}\right)^{\frac{9}{2}}}-1\right\}\frac{1}{10^5};$$

$$\text{or }-\frac{7}{156}\left\{\frac{1}{10^5}-\frac{1}{11^5}\cdot\left(\frac{11}{10}\right)^{\frac{1}{2}}\right\},$$

which is evidently, in *numerical* value, less than $\frac{1}{20}\left(\frac{1}{10^5} - \frac{1}{11^5}\right)$ or $\frac{.000004\ldots}{20}$ or $.0000002\ldots$. Hence we have

$$\sqrt{1+\frac{1}{10}} = 1 + \frac{1}{2}\frac{1}{10} - \frac{1}{2^3}\frac{1}{10^2} + \frac{1}{2^4}\frac{1}{10^3} - \frac{5}{2^7}\frac{1}{10^4} + \frac{7}{2^8}\frac{1}{10^5}$$

$$+ \text{ an error between } 0 \text{ and } -.0000002\ldots.$$

Ex. 3. Let $f(x) = e^x$, $a = 0$, $h = 1$, $n = 10$; then

$$f^n(x) = e^x; \quad \therefore f^{10}(x) - f(0) = e^x - 1:$$

the greatest and least values of this for all values of x between 0 and 1 are $e - 1$ and 0. Hence the error lies between $\frac{e-1}{\Gamma(10)}$ and 0, or between $\frac{2}{\Gamma(10)}$ and 0 since e is < 3; therefore

$$e = 1 + \frac{1}{\Gamma 1} + \frac{1}{\Gamma 2} + \frac{1}{\Gamma 3} \cdots \frac{1}{\Gamma(10)}$$

+ an error between 0 and $\frac{2}{\Gamma(10)}$, which has no significant digit in the first 6 decimal places.

Condition of convergency.

†† 137. Let u_n denote the *numerical* value of

$$f^n(x)\frac{(x-a)^n}{\Gamma n},$$

for any value of x between a and $a+h$; then if the limiting value of u_n when n approaches ∞ is zero, the developement (1) is converging, and we may assert that

$$f(a+h) = f(a) + f^1(a)\frac{h}{\Gamma 1} + f^2(a)\frac{h^2}{\Gamma 2}$$

$$+ f^3(a)\frac{h^3}{\Gamma 3} \ldots \textit{ ad infinitum.}$$

For then it is clear that

$$\{f^n(x) - f^n(a)\} \frac{(x-a)^n}{\Gamma n},$$

and therefore $M\frac{h^n}{\Gamma n}$, $N\frac{h^n}{\Gamma n}$, and therefore $Q\frac{h^n}{\Gamma n}$ may be diminished *ad libitum* by sufficiently increasing n: therefore by (130) the developement is converging, and we may assert that

$$f(a+h) = f(a) + f^1(a)\frac{h}{\Gamma 1} + f^2(a)\frac{h^2}{\Gamma 2}$$

$$+ f^3(a)\frac{h^3}{\Gamma 3} \dots \text{ad infinitum.}$$

††138. If, for all values of n greater than a certain value r, $\frac{u_{n+1}}{u_n}$ never exceeds a certain ratio a which is less than unity, the developement is converging. **A simpler condition of convergency.**

For then we have $\frac{u_{r+1}}{u_r}$ not $> a$, $\frac{u_{r+2}}{u_{r+1}}$ not $> a$, $\frac{u_{r+3}}{u_{r+2}}$ not $> a \dots\dots \frac{u_n}{u_{n-1}}$ not $> a$; and therefore, multiplying these inequalities, we have $\frac{u_n}{u_r}$ not $> a^{n-r}$, or u_n not $> a^{n-r}u_r$. Now since a is less than unity, the limiting value of a^{n-r} when n approaches ∞ is zero, and therefore, zero is also the limiting value of $a^{n-r}u_r$, and therefore of u_n: consequently the developement is converging.

139. By means of Lemma XXI. we arrive at the conclusion, that there necessarily exists a perfect developement of $f(a+h)$ in the form **Taylor's series arrived at somewhat differently.**

$$A_0 + A_1 h + A_2 h^2 \dots + A_n h^n + R,$$

provided none of the quantities $f(a)$, $f^1(a)$, $f^2(a) \dots f^n(a)$ be infinite: we shall now prove the same thing somewhat differently.

By (131) we may assume that

$$f(x) = A_0 + A_1(x-a) + A_2(x-a)^2 \dots + A_n(x-a)^n + R \dots\dots (1),$$

$A_0, A_1 \dots A_n$ being any arbitrary constants, and R that quantity, whatever it be, which must be added to the second member of the equation to make it equal to the first. If we can so determine $A_0, A_1, A_2 \dots A_n$ that the limiting value of $\dfrac{R}{(x-a)^n}$ when x approaches a shall be zero, then the developement (1) will be a perfect developement. Now if this limiting value $= 0$, it follows from Lemma XX. Cor. 3, that

$$\frac{dR}{dx}, \ \frac{d^2R}{dx^2}, \ \frac{d^3R}{dx^3} \dots \frac{d^nR}{dx^n}$$

must be each zero when $x = a$. Hence, differentiating (1) put in the form

$$f(x) - A_0 - A_1(x-a) - A_2(x-a) \dots\dots - A_n(x-a) = R$$

n times successively, and then putting $x = a$, we obtain the following equations:

$$\left.\begin{array}{l} f(a) - \quad A_0 = 0 \\ f'(a) - \Gamma 1 . A_1 = 0 \\ f^2(a) - \Gamma 2 . A_2 = 0 \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ f^n(a) - \Gamma n . A_n = 0 \end{array}\right\} \dots\dots\dots (2),$$

which equations give us the values of $A_0, A_1, A_2 \dots A_n$, necessary to make (1) a perfect developement.

And conversely, if we give the values (2) to $A_0, A_1, A_2 \dots A_n$, we shall evidently have

$$R = 0, \ \frac{dR}{dx} = 0, \ \frac{d^2R}{dx^2} = 0 \dots \frac{d^nR}{dx^n} = 0, \text{ when } x = a;$$

and therefore the limiting value of $\dfrac{R}{(x-a)^n}$ when x approaches a, will (by Lemma XX. Cor. 2.) be zero; i.e. the developement will be perfect.

Hence it appears (putting $x - a = h$) that

$$f(a) + f'(a)\frac{h}{\Gamma 1} + f^2(a)\frac{h^2}{1.2} \ldots\ldots f^n(a)\frac{h^n}{\Gamma n} + R$$

is a perfect developement, and the only perfect developement, of $f(a+h)$.

We must except however the case where any of the quantities $f(a)$, $f'(a)$, $f^2(a) \ldots f^n(a)$ are infinite, for then we cannot be sure that the equations (2) are satisfied by giving A_0, A_1, $A_2 \ldots$ &c. the above values; for instance, if $f^3(a) = \frac{1}{0}$, we cannot assert that $f^3(a) - \Gamma 3 . A_3 = 0$, if we give A_3 the value $\frac{1}{0}$, for then the quantity $f^3(a) - \Gamma 3 . A_3$ assumes the form $\frac{1}{0} - \frac{1}{0}$, which we cannot assert is zero.

140. If $f(x)$, and all its differential coefficients below a certain one, the p^{th} suppose, be finite, but the p^{th} (and all above it by Lemma XXIII.) infinite; then the developement (1) holds as a perfect developement for all values of n less than p, but for all other values of n it fails.

The failure of Taylor's series.

141. When a failure of this kind takes place, it is generally possible to obtain a perfect developement containing fractional powers of h after the p^{th} term.

A differential coefficient becoming infinite when $x = a$, indicates the appearance of fractional powers in the developement.

For let $$F(x) = f(x) - f(a) - f'(a)\frac{x-a}{\Gamma 1}$$

$$- f^2(a)\frac{(x-a)^2}{\Gamma 2} \ldots - f^{p-1}(a)\frac{(x-a)^{p-1}}{\Gamma(p-1)}, \text{ see (123)}$$

then it is clear that

$$F(a) = 0, \; F'(a) = 0, \; F^2(a) = 0 \ldots\ldots F^{p-1}(a) = 0,$$

but $F^p(a) = f^p(a) = \infty$; since we suppose that $f(a)$, $f'(a)$, $f^2(a) \ldots\ldots f^{p-1}(a)$ are finite quantities, but $f^p(a)$, $f^{p+1}(a) \ldots$ infinite.

Hence, by Lemma XX. Cor. 2, the limiting value of $\frac{F(x)}{(x-a)^{p-1}}$ is zero, and that of $\frac{F(x)}{(x-a)^p}$ is infinite, when x approaches a.

Now if any function $\phi(x) = 0$ when $x = a$, we may generally find some power of $x - a$, $(x-a)^m$ suppose, such that the limiting value of $\frac{\phi(x)}{(x-a)^m}$ when x approaches a, is neither zero nor infinity. Let us then suppose $(x-a)^m$ to be such a power of $x - a$, that the limiting value of $\frac{F(x)}{(x-a)^m}$ when x approaches a is some constant C, which is neither zero nor infinity: then, since

$$\frac{F(x)}{(x-a)^{p-1}} = \frac{F(x)}{(x-a)^m}.(x-a)^{m-p+1}$$

$$\text{and } \frac{F(x)}{(x-a)^p} = \frac{F(x)}{(x-a)^m}(x-a)^{m-p},$$

it is clear that the limiting values of $\frac{F(x)}{(x-a)^{p-1}}$ and $\frac{F(x)}{(x-a)^p}$ are $C.0^{m-p+1}$ and $C.0^{m-p}$. If m be less than $p-1$ the former of these is infinite, which we know not to be the case; and if m be greater than p the latter will be zero, which we know not to be the case: hence m lies between $p-1$ and p, and is therefore a fraction.

If therefore we assume $\frac{F(x)}{(x-a)^m} = C + Q$, Q will be some quantity which becomes zero when $x = 0$, and we have, putting for $F(x)$ its value

$$f(x) = f(a) + f'(a)\frac{(x-a)}{\Gamma 1} + f^2(a)\frac{(x-a)^2}{\Gamma 1}\dots\dots$$

$$f^{p-1}(a)\frac{(x-a)^{p-1}}{\Gamma(p-1)} + (C+Q)(x-a)^m,$$

which is a perfect developement containing a fractional power $C(x-a)^m$, m being greater than $p-1$, and less than p.

Thus it appears that when any differential coefficient $f^r(a)$ becomes infinite, it indicates the appearance of a fractional power in the developement of $f(x)$ in powers of $x - a$.

142. We may generally apply Taylor's series, in the following manner, to determine the developement of $f(x)$, when fractional powers appear in it in consequence of some of its differential coefficients becoming infinite when $x = a$. How the developement is to be obtained when any of the differential coefficients become infinite.

Put $x = a + z^r$, and $f(x)$ will become a function of z, $\phi(z)$ suppose: then, if possible, so determine r that neither $\phi(z)$ nor any of its differential coefficients shall become infinite when $z = 0$, and this being the case we shall have, by Taylor's Theorem,

$$\phi(z) = \phi(0) + \phi'(0)\frac{z}{\Gamma 1} + \phi^2(0)\frac{z^2}{\Gamma 2} \dots \phi^n(0)\frac{z^n}{\Gamma n} + R,$$

n being as large as we please.

Now here put for z its value $(x-a)^{\frac{1}{r}}$, and for $\phi(z)$ its value $f(x)$, and we have

$$f(x) = \phi(0) + \phi'(0)\frac{(x-a)^{\frac{1}{r}}}{\Gamma 1}$$
$$+ \phi^2(0)\frac{(x-a)^{\frac{2}{r}}}{\Gamma 2} \dots \phi^n(0)\frac{(x-a)^{\frac{n}{r}}}{\Gamma n} + R,$$

which is the developement required containing fractional powers.

There are also other ways of substituting for x, so as to obtain the developement of $f(x)$ by Taylor's series.

Ex. Let $f(x) = \sin\{x + 1 + (x-1)^{\frac{3}{2}}\}$; Example.

here it is easy to see that $f^2(0) = \infty$; therefore we shall not be able to develope $f(x)$ in this case in integral powers of $x - 1$. To obtain the developement by means of Taylor's Theorem, put $x = 1 + z^2$, and then

$$f(x) = \sin(2 + z^2 + z^3) = \phi(z),$$

and it is easy to see that neither $\phi(z)$ nor any of its differential coefficients become infinite when $z = 0$: therefore we have

$$f(x) = \phi(0) + \phi'(0)\frac{(x-1)^{\frac{1}{2}}}{\Gamma 1} + \phi^2(0)\frac{(x-1)}{\Gamma 2} + \phi^3(0)\frac{(x-1)^{\frac{3}{2}}}{\Gamma 3} + \&c.$$

By actual differentiation, and putting $z = 0$, we find

$$\phi(0) = \sin 2, \;\; \phi^1(0) = 0, \;\; \phi^2(0) = 2\cos 2, \;\; \phi^3(0) = 6\cos 2 \ldots \&c.$$

and therefore,

$$\sin\left\{x + 1 + (x-1)^{\frac{3}{2}}\right\} = \sin 2 + \cos 2 . (x-1) + \cos 2 . (x-1)^{\frac{3}{2}} + \&c.$$

Same example done more simply.

This method however is not generally the simplest in practice; other substitutions for x often bring out the result more readily. Thus assume

$$x - 1 + (x-1)^{\frac{3}{2}} = z,$$

$$\text{and then } f(x) = \sin(2 + z) = \phi(z), \text{ suppose;}$$

and it is evident that neither $\phi(z)$ nor any of its differential coefficients become infinite when $x = 1$; i. e. when $z = 0$; we have therefore

$$\phi(z) = \phi(0) + \phi'(0)\frac{z}{\Gamma 1} + \phi^2(0)\frac{z^2}{\Gamma 2} + \&c.$$

Now since $\phi(z) = \sin(2 + z)$, we have

$$\phi(0) = \sin 2, \quad \phi'(0) = \cos 2, \quad \phi^2(0) = -\sin 2,$$
$$\phi^3(0) = -\cos 2 \ldots \&c.;$$

and therefore, restoring for z its value, we have

$$f(x) = \sin 2 + \cos 2 . \left\{(x-1) + (x-1)^{\frac{3}{2}}\right\}$$
$$- \sin 2 . \left\{(x-1) + (x-1)^{\frac{3}{2}}\right\}^2 \ldots\ldots$$

if we arrange this in powers of $(x - 1)$, expanding each term by the binomial theorem, we find the developement required.

$f(a)$ being infinite indicates the

143. These methods clearly suppose that $f(a)$ is not infinite.

If $f(a) = \infty$, then if we assume

$$(x-a)^s f(x) = \phi(x),$$

appearance of negative powers in the series.

and so determine s that $\phi(a)$ shall not be infinite, we may develope $\phi(x)$ as above in the form

$$\phi(x) = A_0 + A_1(x-a)^{\frac{1}{r}} + A_2(x-a)^{\frac{2}{r}} \ldots \&c.$$

and therefore, restoring $f(x)$, we have

$$f(x) = A_0(x-a)^{-s} + A_1(x-a)^{-s+\frac{1}{r}} \ldots \&c.$$

Hence, $f(x)$ becoming infinite when $x = a$, indicates the appearance of negative powers in the series. (For examples, see Appendix H.)

†† 144. *Suppose we have a relation between* x *and* y, *viz.* f (x y) = 0, *to develope* y *in a perfect series of powers of* x.

Developement of a function given by an equation.

To do this all that is necessary in general is, to write down to the equation $f(xy)$ and its successive derivatives, and then to put $x = 0$, and so find the values of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c. when $x = 0$, and thus obtain the perfect developement of y in powers of x by Taylor's Theorem. An example will best explain this method.

Let the relation between x and y be

Example.

$$y^3 - xy + 2x = 1 \ldots (1);$$

then differentiating successively, we have

$$(3y^2 - x)p - y + 2 = 0 \ldots (2),$$

$$(3y^2 - x)q + 6yp^2 - 2p = 0 \ldots (3),$$

when $p, q, r \ldots$ are, for brevity, put for $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} \ldots$

$$(3y^2 - x)r + (18yp - 3)q + 6p^3 = 0 \ldots (4)$$

....

&c. &c.

Hence putting $x = 0$, we find

from (1) $y^3 = 1$; $\therefore y = 1$, (at least 1 is one of the 3 values of y);

$$\therefore \text{ from (2) } \quad 3p + 1 = 0; \quad \therefore p = -\frac{1}{3};$$

$$\therefore \text{ from (3) } \quad 3q + \frac{6}{9} + \frac{2}{3} = 0; \quad \therefore q = -\frac{4}{9};$$

$$\therefore \text{ from (4) } \quad 3r + 9 . \frac{4}{9} + \frac{2}{9} = 0; \quad \therefore r = -\frac{38}{27};$$

....

&c. &c.

Hence by Taylor's series, we have

$$y = 1 - \frac{1}{3}\frac{x}{\Gamma 1} - \frac{4}{9} . \frac{x^2}{\Gamma 2} - \frac{38}{27}\frac{x^3}{\Gamma 3} \dots \text{ \&c.}$$

How we must proceed when Taylor's series fails.

†† 145. It often happens however that we thus get infinite values for some of the quantities p, q, r, &c., and of course this method fails; we must then proceed as follows.

Let a be a value (or one of the values) of y when $x = 0$, then substitute $a + ux^m$ for y in the given equation $f(xy) = 0$, and, if possible, give m such a value, that the limiting value of u when x approaches 0 shall not be 0 or ∞, but some quantity, b suppose. Then assume $u = b + vx^n$, substitute this value of u in the equation and so determine n, if possible, that the limiting value of v when x approaches zero shall not be 0 or ∞, but some quantity, c suppose. Then assume $v = c + wx^p$; it is clear that in this manner we obtain a perfect developement for y, viz.,

$$y = a + bx^m + cx^{m+n} + R, \quad \text{putting } R = wx^{m+n+p},$$

which we may carry as far as we please.

Several different series

It generally happens that we obtain more than one value of y when $x = 0$, and more than one value of m, or of n,

or of r, &c., which satisfies the above conditions, in which case we find more than one perfect developement for y. generally obtained in this manner.

An example will be necessary to make the nature of this process of developement clear. Example.

Let $y^3 - 3axy + x^3 = 0$,

then $y = 0$ when $x = 0$; assume $\therefore$ $y = ux^m$,

and $\therefore$ substituting

$$u^3 x^{3m} - 3aux^{m+1} + x^3 = 0 \ldots (1).$$

Now here the powers of x cannot be all different, since the coefficients of them are, either constant quantities not equal to zero, or quantities which we suppose not to approach zero as their limiting value when x approaches zero: for it is clear that if all the powers of x be different, and if the equation be divided by the lowest power, and x then put equal to zero, the coefficient of the lowest power must necessarily become zero, contrary to hypothesis.

Since then the powers of x cannot be all different some two of them must be the same; either the first and second, or the first and third, or the second and third.

(1) Suppose that the first and third are the same; then $3m = m + 1$, and $\therefore$ $m = \frac{1}{2}$; then (1) becomes, dividing out $x^{\frac{3}{2}}$,

$$u^3 - 3au + \text{a positive power of } x = 0,$$

which gives $u = 0$ or $\pm\sqrt{3a}$ when $x = 0$: we must of course reject $u = 0$, since we want only those values of u which are neither 0 nor ∞.

(2) Suppose that $3m = 3$, and $\therefore$ $m = 1$, then we have

$$u^3 + 1 + \text{a negative power of } x = 0,$$

which $\therefore$ gives $u = \infty$ when $x = 0$; this value of m must therefore be rejected.

(3) Suppose that $m+1=3$, and $\therefore m=2$; then we have

$$-3au + 1 + \text{a positive power of } x = 0,$$

$$\text{which gives } u = \frac{1}{3a} \text{ when } x = 0.$$

Hence it appears that there are two values of m, namely $\frac{1}{2}$ and 2, which answer our purpose, and that when x approaches 0, u in the former case approaches $\sqrt{3a}$ or $-\sqrt{3a}$ as its limiting value, and in the latter case $\frac{1}{3a}$.

Hence, proceeding only as far as the first terms, we have the perfect developements

$$y = \pm\sqrt{3a}\, x^{\frac{1}{2}} + R,$$

$$\text{and } y = \frac{x^2}{3a} + R'.$$

To obtain the second terms of these developements we must assume $m = \frac{1}{2}$, or 2, and

$$u = \pm\sqrt{3a} + vx^n, \quad \text{or } u = \frac{1}{3a} + vx^n \text{ respectively,}$$

and substitute in the equation (1) and proceed as before, and so we may go on to any number of terms. It is often useful to obtain the first terms of the developements as in this Example, but we seldom have occasion to go any farther.

The method stated in general. It is clear from this Example, that to determine the first terms of the developements, we have only to put $y = a + ux^m$ (a being the value of y when $x = 0$) in the given equation, then suppose the powers of x to be the same, two by two. If, when we suppose two powers the same, any of the remaining powers is lower than them, then there is no value of u such as we want corresponding to these two powers, (as in the case where we assumed $3m = 3$ in the Example). If, however, none of the remaining powers is lower than the two supposed to be the same; then dividing out the lowest power of x from the equation, and putting $x = 0$, we obtain a value or values

of u, (rejecting of course any value which = zero); and thus we obtain the developements required.

(For examples and certain simplifications see Appendix I.)

†† 146. *If* f(xy) *be any function of* x *and* y, *and if we put* x = a + h, y = b + k, *to develope* f(xy) *in powers of* h *and* k. Developement of a function of two variables $f(a+h, b+k)$ in powers of h and k.

Putting $a + h$ for x, we have, by Taylor's theorem,

$$f(xy) = f(a+h, y) = f(ay) + d_a f(ay)\frac{h}{\Gamma 1}\ldots$$
$$+ d_a^m f(ay)\cdot\frac{h^m}{\Gamma m} + \&c.\ldots(1),$$

but by the same theorem, if we put $b + k$ for y, we have

$$f(ay) = f(a, b+k) = f(ab) + d_b f(ab)\frac{k}{\Gamma 1}\ldots$$
$$+ d_b^n f(ab)\frac{k^n}{\Gamma n}\ldots\&c.\ldots(2).$$

Substituting this value of $f(ay)$ in each term of (1), we obtain the developement required.

Now the general term of (1) is $d_a^m f(ay)\dfrac{h^m}{\Gamma m}$, and by (2) the general term of this is

$$d_a^m\left\{d_b^n f(ab)\frac{k^n}{\Gamma n}\right\}\frac{h^m}{\Gamma m}, \quad \text{or } d_a^m d_b^n f(ab)\cdot\frac{h^m k^n}{\Gamma m\Gamma n},$$

hence, giving m and n their several values, we obtain

$$f(a+h, b+k) = f(ab) + d_a f(ab)\frac{h}{\Gamma 1} + d_b f(ab)\frac{k}{\Gamma 1}$$
$$+ d_a^2 f(ab)\frac{h^2}{\Gamma 2} + d_a d_b f(ab)\frac{hk}{\Gamma 1\Gamma 1} + d_b^2 f(ab)\frac{k^2}{\Gamma 2}$$
$$+ d_a^3 f(ab)\frac{h^3}{\Gamma 3}\ldots\ldots\ldots\ldots\ldots\ldots\&c.\ldots\ldots\&c.$$
$$\ldots\ldots\ldots\&c.\ldots\ldots\ldots\&c.\ldots\ldots\ldots\ldots$$

Symbolical form of the developement and of Taylor's Series.

If we use a symbolical form similar to that in 110, this expansion may be expressed in the following manner, {observing that $\frac{\Gamma n}{\Gamma r\,\Gamma(n-r)}$ is the coefficient of $h^r k^{n-r}$ in $(h+k)^n$ expanded}, viz:

$$f(a+h, b+k) = \left\{1 + \frac{(d_a h + d_b k)^1}{\Gamma 1} + \frac{(d_a h + d_b k)^2}{\Gamma 2} \ldots \&c.\right\} f(ab)$$

$$= \epsilon^{d_a h + d_b k} f(ab).$$

Taylor's series expressed similarly may be thus written

$$f(a+h) = \epsilon^{fh}.\, a,$$

$f^0(a)$ being supposed to be the same as $f(a)$.

Another proof of the result that $d_x{}^m d_y{}^n y = d_y{}^n d_x{}^m y$.

†† 147. In the expansion of $f(a+h, b+k)$ above obtained, the coefficient of $\frac{h^m k^n}{\Gamma m\,\Gamma n}$ is $d_a^m d_b^n f(ab)$. Now if we had arrived at this developement by first putting $b+k$ for y and expanding, and then $a+h$ for x and expanding again, it is easy to see that the coefficient of $\frac{h^m k^n}{\Gamma m\,\Gamma n}$ would have been $d_b^n d_a^m f(ab)$, hence we must have

$$d_b^n d_a^m f(ab) = (d_a^m d_b^n f(ab),$$

a result which we obtained before in (106).

CHAPTER X.

DETERMINATION OF THE LIMITING VALUES OF VANISHING FRACTIONS.

148. THE Differential Calculus may often be employed with great advantage to determine the limiting values of vanishing fractions; i.e. functions which assume the illusory form $\frac{0}{0}$ when the variable receives some particular value. Vanishing Fractions.

If $f(a)=0$ *and* $\phi(a)=0$, *to determine the limiting value of* $\frac{f(x)}{\phi(x)}$ *when* x *approaches* a. To determine the limiting value of a vanishing fraction.

Since $f(a)$ and $\phi(a)$ are zero, $\frac{f(x)}{\phi(x)}=\frac{f(x)-f(a)}{\phi(x)-\phi(a)}$; hence by Lemma XIX. the limiting value of $\frac{f(x)}{\phi(x)}$ is $\frac{f'(a)}{\phi'(a)}$; or more generally, by Lemma XX. the limiting value of $\frac{f'(x)}{\phi'(x)}$ when x approaches a is also the limiting value of $\frac{f(x)}{\phi(x)}$; if it should so happen that $\frac{f'(a)}{\phi'(a)}$ is not an illusory quantity, then it is the limiting value we seek: but if $f'(a)=0$ and $\phi'(a)=0$, then by the same Lemma, Cor. 1, the limiting value of $\frac{f^2(x)}{\phi^2(x)}$ is that of $\frac{f(x)}{\phi(x)}$, and therefore $\frac{f^2(a)}{\phi^2(a)}$, if not illusory, is the value we seek: if however $f^2(a)$ and $\phi^2(a)$ be both zero, then we must try the third differential coefficients, and so on till we come to two differential coefficients which do not both vanish (nor of course become both infinite) when $x=a$; and thus we shall obtain the limiting value we seek.

Case in which this method fails.

If we come to two differential coefficients which both become infinite when $x = a$ then there is no use in going farther, since all the succeeding differential coefficients will be infinite also by Lemma XXIII.

How we are to proceed in such a case.

149. In such a case we must proceed as follows.

Put $a + h$ for x and expand $f(a+h)$ in a perfect series $f(a)+Ah^{\alpha}+R'$ and $\phi(a+h)$ in a perfect series $\phi(a)+Bh^{\beta}+R$; then since $f(a)$ and $\phi(a)$ are each zero, we have

$$\frac{f(a+h)}{\phi(a+h)} = \frac{Ah^{\alpha}+R}{Bh^{\beta}+R'} = h^{\alpha-\beta}\cdot\frac{A+\dfrac{R}{h^{\alpha}}}{B+\dfrac{R'}{h^{\beta}}}.$$

Hence, since the series are perfect and therefore the limiting values of $\dfrac{R}{h^{\alpha}}$ and $\dfrac{R'}{h^{\beta}}$ when h approaches 0 each zero, the limiting value of $\dfrac{f(a+h)}{\phi(a+h)}$ when h approaches zero is the same as that of $h^{\alpha-\beta}\cdot\dfrac{A}{B}$. Now if $\alpha > \beta$ this $= 0$; if $\alpha < \beta$ it $= \infty$; and if $\alpha = \beta$ it $= \dfrac{A}{B}$. Thus the required limiting value is found.

This latter method is sometimes preferable in cases where the former does not fail.

Examples of vanishing fractions.

Let $$\frac{f(x)}{\phi(x)} = \frac{x^n - a^n}{x^3 - ax^2 + a^2x - a^3},$$

which assumes the form $\dfrac{0}{0}$ when $x = a$;

$$\text{here } f'(x) = nx^{n-1} = na^{n-1} \text{ when } x = a,$$
$$\phi'(x) = 3x^2 - 2ax + a^2 = 2a^2 \text{ when } x = a;$$

hence the limiting value of $\dfrac{f(x)}{\phi(x)}$ when x approaches a is $\dfrac{f'(a)}{\phi'(a)}$ or $\dfrac{n}{2}a^{n-3}$.

Example 2.

Let $\dfrac{f(x)}{\phi(x)} = \dfrac{\tan x - \sin x}{x^3}$, which assumes the form $\dfrac{0}{0}$ when $x = 0$,

$$\frac{f'(x)}{\phi'(x)} = \frac{\sec^2 x - \cos x}{3x^2} = \frac{0}{0} \text{ when } x = 0$$

$$= \frac{1}{\cos^2 x} \cdot \frac{1 - \cos^3 x}{3x^2}.$$

Important simplification we may often make in the process.

150. Here we must make a remark of some importance. Suppose that we obtain $\dfrac{f'(x)}{\phi'(x)}$ in the form UV, and that we know A to be the limiting or actual value of U corresponding to the value of x which makes the fraction vanish; then, by Lemma VIII., the limiting value we require is $A \times$ limiting value of V. Hence if in the process of obtaining the limiting value of a vanishing fraction, we find that $\dfrac{f'(x)}{\phi'(x)}$ has any factor whose limiting or actual value we know, we may always substitute immediately for that factor its value whatever it be, and so simplify the operation.

Thus in the present example the factor $\dfrac{1}{\cos^2 x} = 1$ when $x = 0$, we may therefore substitute 1 for it, and then we have only to find the limiting value of $\dfrac{1 - \cos^3 x}{3x^2}$; i.e. of

$$\frac{d(1 - \cos^3 x)}{d(3x^2)} \text{ or } \frac{\cos^2 x \sin x}{2x},$$

and here again putting 1 instead of the factor $\cos^2 x$ we have only to find the limiting value of $\dfrac{\sin x}{2x}$; i.e. of $\dfrac{d \sin x}{d\, 2x}$ or $\dfrac{\cos x}{2}$, which is $\frac{1}{2}$. Hence the limiting value required is $\frac{1}{2}$.

Example 3. Let $\dfrac{f(x)}{\phi(x)} = a^{n-1}.\dfrac{a-(a^n-x^n)^{\frac{1}{n}}}{x^n}$, which assumes the form $\dfrac{0}{0}$ when $x=0$,

$$\frac{f'(x)}{\phi'(x)} = a^{n-1}.\frac{x^{n-1}(a^n-x^n)^{\frac{1}{n}-1}}{nx^{n-1}} = \frac{a^{n-1}}{n}(a^n-x^n)^{\frac{1}{n}-1},$$

the limiting value of which when $x=0$ is evidently $\dfrac{1}{n}$.

Example 4. Let $\dfrac{f(x)}{\phi(x)} = (1-x)\tan\dfrac{\pi x}{2}$, which assumes the illusory form $0.\infty$ when $x=1$. This is the same thing as the form $\dfrac{0}{0}$, as appears if we put $\dfrac{f(x)}{\phi(x)}$ in the form

$$\frac{(1-x)\sin\dfrac{\pi x}{2}}{\cos\dfrac{\pi x}{2}}.$$

Now the limiting value of this is the same as that of

$$\frac{1-x}{\cos\dfrac{\pi x}{2}} \text{ (since } \sin\frac{\pi x}{2} = 1 \text{ when } x=1);$$

$$\text{i. e. of } \frac{-1}{-\dfrac{\pi}{2}\sin\dfrac{\pi x}{2}} \text{ which } = \frac{2}{\pi}.$$

The illusory form $\infty - \infty$ is the same as the form $\frac{0}{0}$.

151. A function sometimes assumes the form $\infty - \infty$, which is the same form as $\dfrac{0}{0}$: for let $\dfrac{1}{f(x)} - \dfrac{1}{\phi(x)}$ be a function which assumes the form $\infty - \infty$ when $x=a$, in consequence of $f(a)$ and $\phi(a)$ being each zero: then

$$\frac{1}{f(x)} - \frac{1}{\phi(x)} = \frac{\phi(x)-f(x)}{f(x)\,\phi(x)},$$

which assumes the form $\frac{0}{0}$ when $x = 0$; thus the illusory forms $\frac{0}{0}$ and $\infty - \infty$ are identical.

Hence when we wish to find the limiting value of a function which assumes the form $\infty - \infty$, we must reduce it to a fraction, so that it shall assume the form $\frac{0}{0}$, and then proceed as above. How to find the limiting value of a function when it becomes $\infty - \infty$.

$\frac{2}{x^2 - 1} - \frac{1}{x - 1}$ becomes $\infty - \infty$ when $x = 1$; to find its limiting value we must put it in the form $\frac{2 - (x + 1)}{x^2 - 1}$, which assumes the form $\frac{0}{0}$, and whose limiting value therefore is that of $\frac{-1}{2x}$, or $-\frac{1}{2}$. Example 5.

152. Let $\frac{f(x)}{\phi(x)} = \frac{x - a + \sqrt{2ax - 2a^2}}{\sqrt{x^2 - a^2}}$, which assumes the form $\frac{0}{0}$ when $x = a$. Example 6. Case of failure of common method.

Here $f'(a)$ and $\phi'(a)$ are both infinite, we must therefore put $x = a + h$, and then $\frac{f(x)}{\phi(x)}$ becomes $\frac{h + \sqrt{2ah}}{\sqrt{h(2a + h)}}$, which $= \frac{h^{\frac{1}{2}} + \sqrt{2a}}{\sqrt{2a + h}}$, which $= 1$ when $h = 0$, hence the required limiting value is 1.

$$\text{Let } \frac{f(x)}{\phi(x)} = \frac{\sin^3\sqrt{x^2 - 1} + (x - 1)^2}{(x^2 - 1)\sqrt{x - 1}},$$ Example 7. A similar case.

which assumes the form $\frac{0}{0}$ when $x = 1$; here $f^2(1)$ and $\phi^2(1)$

are both infinite; we must therefore put $x = 1 + h$, and then $\dfrac{f(x)}{\phi(x)}$ becomes

$$\frac{\sin^3 \sqrt{h(h+2)} + h^3}{h(h+2)h^{\frac{1}{2}}}.$$

Now $\quad \sin z = z - \dfrac{z^3}{6}$ &c....

$\sin^3 z = z^3 - \dfrac{z^5}{2}$ &c....

$\therefore$ putting $z^3 = [\{h(h+2)\}]^{\frac{3}{2}} = 2^{\frac{3}{2}} h^{\frac{3}{2}} +$ &c....

we find $f(x) = h^{\frac{3}{2}} + R$ a perfect developement,

and $\phi(x) = 2h^{\frac{3}{2}} + R'$ ditto,

and $\therefore$ the required limiting value is that of $\dfrac{2^{\frac{3}{2}} h^{\frac{3}{2}}}{2h^{\frac{3}{2}}}$ which $= \sqrt{2}$. (See Appendix J.)

When we are finding $\frac{dy}{dx}$ from an equation it sometimes assumes the form $\frac{0}{0}$.

††153. It often happens, when we are finding $\dfrac{dy}{dx}$ from an equation between y and x, as in 145, that the result comes out in the form $\dfrac{0}{0}$, which of course leaves us in ignorance as to what the true value of $\dfrac{dy}{dx}$ is. Thus suppose that we wish to obtain the value of $\dfrac{dy}{dx}$ when $x = 0$ from the equation,

$$x^4 + 3a^2x^2 - 4a^2xy - a^2y^2 \text{ .... } (1).$$

Differentiating, we have

$$4x^3 + 6a^2x - 4a^2y - (4a^2x + 2a^2y)\,p = 0 \text{ ...... } (2) \left(p = \frac{dy}{dx}\right).$$

Now putting $x = 0$, and $\therefore y = 0$ in virtue of (1), we have

$$0 - 0 \,.\, p = 0, \text{ or } p = \frac{0}{0};$$

thus when $x = 0$ we cannot find p in this manner.

But differentiating the equation (2) we have

In such a case we must differentiate again in order to find $\frac{dy}{dx}$.

$$6x^2 + 3a^2 - 2a^2p - (2a^2 + a^2p)\,p - (2a^2x + a^2y)\,q = 0 \quad \left(q = \frac{d^2y}{dx^2}\right),$$

and putting $x = 0$, and $\therefore\ y = 0$, here we find

$$3a^2 - 2a^2p - (2a^2 + a^2p)\,p = 0,$$

$$\text{or } p^2 + 4p - 3 = 0\,;$$

$$\therefore\ p = -2 \pm \sqrt{4+3},$$

and thus by differentiating twice we obtain p, and we find that it has two values $-2 + \sqrt{7}$ and $-2 - \sqrt{7}$.

As another example, suppose that we wish to obtain p when $x = 0$ from the equation

Another example.

$$x^4 + ay^3 - 2axy^2 - 3ax^2y = 0 \ldots\ldots\ldots (1),$$

differentiating we find

$$(4x^3 - 2ay^2 - 6axy) + (3ay^2 - 4axy - 3ax^2)\,p = 0 \ldots\ldots (2),$$

here put $x = 0$, and $\therefore\ y = 0$ in virtue of (1), and we find $p = \dfrac{0}{0}$.

But differentiating again we have

$$\frac{12x^2}{a} - 6y - 4(2y + 3x)\,p + 2(3y - 2x)p^2 + (3y^2 - 4xy - 3x^2)q = 0,$$

here put $x = 0$, and $\therefore\ y = 0$, and we find again $p = \dfrac{0}{0}$.

Differentiating again, therefore, we have

$$0 = -18p - 12p^2 + 6p^3 + \text{terms multiplied by } x \text{ or } y,$$

here putting $x = 0$ and $y = 0$, we find

$$p^3 - 2p^2 - 3p = 0,$$

which gives $p = 0$ or $p = 1 \pm \sqrt{1+3} = 3$ or -1.

And thus by differentiating three times we obtain p, and we find that it has three values 0, 3 and -1.

Thus it appears that where the first differentiation of an equation fails to give us $\frac{dy}{dx}$ for a particular value of x, in consequence of all the terms becoming zero, we must differentiate successively until we come to an equation, all the terms of which do not vanish when we give x that particular value.

These differentiations may be performed on the supposition that p is constant.

††154. If we perform these differentiations on the supposition that p is a constant, we shall arrive at a correct result: for it is easy to see from the above examples that all the terms obtained by differentiating p once or oftener vanish when we give x the particular value for which we wish to find p, and therefore do not affect our final equation, which gives us the value of p. Hence we may always differentiate on the supposition that p is constant, and this will somewhat simplify the process.

A somewhat different method of finding p by the rule for vanishing fractions.

††155. The values of p are sometimes found by the method of finding the limiting value of a vanishing fraction given in (148); thus in the first example we have

$$p = \frac{2x^3 + 3a^2x - 2a^2y}{2a^2x + a^2y} = \frac{0}{0} \text{ when } x = 0,$$

therefore by (148),

$$p = \frac{6x^2 + 3a^2 - 2a^2p}{2a^2 + a^2p} = \frac{3 - 2p}{2 + p} \text{ when } x = 0;$$

$$\therefore\ 2p + p^2 = 3 - 2p, \quad p^2 + 4p - 3 = 0,$$

which is the result we arrived at before.

Another example.

†† 156. The following is an example where p assumes the form $\frac{0}{0}$ for other values of x and y besides zero.

To find p when $x = a$ from the following equation, viz.

$$ay^2 - 2a^2y - 2x^3 + 3ax^2 = 0 \ldots\ldots (1).$$

Differentiating, we have

$$(2ay - 2a^2)\,p - 6x^2 + 6ax = 0.$$

When $x = a$, and therefore $y = a$, in virtue of (1), this equation gives $p = \frac{0}{0}$; therefore, differentiating again, considering p constant, we have

$$2ap^2 - 12x + 6a = 0;$$

which, when $x = a$, gives

$$p^2 - 3 = 0; \quad \therefore p = \pm\sqrt{3}.$$

†† 157. There are two objections to this method of finding the values of p. 1st. It is generally very complicated and troublesome when we have to go beyond a second differentiation. 2nd. We have no right to assume that the terms containing $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} \dots$ vanish; for although the coefficients of these terms vanish, yet, since $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} \dots$ may and often do at the same time become infinite, we cannot tell but that these terms may assume the form $0 . \infty$ or $\frac{0}{0}$; and therefore we cannot assume them to be zero. For instance, in the first example, (153), in the result of the second differentiation we have the term $(2a^2x + a^2y)\,q$, which we assume to be zero because $2ax + a^2y = 0$ when $x = 0$; but we have no right to do this, since we cannot tell whether q is infinite or not. Objections to this method of finding p by successive differentiation.

†† 158. The following method will be found free from this objection, and very simple in practice, especially when it is our object to find p for the values $x = 0$ and $y = 0$. A method free from these objections explained.

(1) Suppose that we wish to find p for the value $x = 0$ from the equation

$$Ax + By + Cx^2 + Dxy + Ey^2 + Fx^3 \dots \&c. = 0 \dots\dots (1).$$

Since $y = 0$ when $x = 0$, it is evident that the value of p we seek is the limiting value of $\frac{y}{x}$ when x approaches zero (see 148): if therefore we put $\frac{y}{x} = u$ and find the limiting value of u, u_0 suppose, when x approaches zero, u_0 is the quantity we wish to determine.

Now put $y = ux$ in (1), divide out x, and we find

$$A + Bu + x(C + Du + Eu^2) + x^2 (F \ldots) \ldots\ldots (2),$$

here let $x = 0$, and then u must become u_0, if therefore u_0 be a finite quantity we have

$$(3) \ldots\ldots A + Bu_0 = 0, \text{ and } \therefore\ u_0 = -\frac{A}{B};$$

which gives u_0.

If it should happen, however, that both A and B are zero, the value we have here obtained for u_0 assumes the form $\frac{0}{0}$, and therefore we cannot thus arrive at the real value of u_0. But then the equation (2), dividing out x, becomes

$$C + Du + Eu^2 + x(F + \&c. \ldots) \ldots = 0;$$

here put $x = 0$, and therefore $u = u_0$, and we find (supposing u_0 not infinite),

$$C + Du_0 + Eu_0^2 = 0;$$

$$\text{and } \therefore\ u_0 = -\frac{D}{2E} \pm \sqrt{\frac{D^2}{4E^2} - \frac{C}{4E}};$$

thus it appears that $\frac{dy}{dx}$ has two values in this case, when $x = 0$.

How the infinite values of u_0 are to be found.

In this process we suppose u_0 to be a finite quantity; therefore, if u_0 admits of infinite values, this method does not give them, and we are left in ignorance as to whether there are such values or not*. But we may easily determine this in the following manner:

Put $x = \frac{y}{u}$ instead of $y = ux$, and the equation (1) becomes (supposing A and B zero)

$$C\frac{1}{u^2} + D\frac{1}{u} + E + y\left(F\frac{1}{u^3} \ldots \&c.\right), \&c. = 0;$$

* In the common method of finding $\frac{dy}{dx}$ the infinite values are sometimes overlooked.

and then putting $y = 0$, we have

$$C\frac{1}{u_0^2} + D\frac{1}{u_0} + E = 0.$$

Hence if E is not zero $\frac{1}{u_0}$ cannot be zero; and therefore u_0 cannot be infinite: but if $E = 0$, then one value of $\frac{1}{u_0}$ is zero, and therefore one value of u_0 infinite: if D also $= 0$, both values of $\frac{1}{u_0}$ are zero, and therefore both values of u_0 infinite.

If C, D, and E be all zero, then in the equation (2), dividing out x^2, and putting $x = 0$, we have

$$F + Gu_0 + Hu_0^2 + Iu_0^3 = 0 \ldots.$$

which equation gives us the finite values of u_0. To obtain the infinite values we have, as before, the equation

$$F\frac{1}{u_0^3} + G\frac{1}{u_0^2} + H\frac{1}{u_0} + I = 0.$$

It appears from these equations that there are in this case 3 values of $\frac{dy}{dx}$ when $x = 0$. If $I = 0$ one of the values of $\frac{1}{u_0}$ is zero, and therefore one of the values of u_0 is infinite: if $I = 0$, $H = 0$, two of the values of $\frac{1}{u_0}$ are zero, and therefore two of u_0 are infinite: if $G = 0$ also, the three values of u_0 are infinite: if $I = 0$ and $F = 0$, a value of u_0 is 0, and a value of $\frac{1}{u_0}$ is 0, and either of the equations becomes $G + Hu_0 = 0$, which gives $u_0 = -\frac{G}{H}$; hence, in this case, the three values of u_0 are 0, ∞, and $-\frac{G}{H}$.

If F, G, H, I, are all zero, we must divide out x^3 in equation (2) and proceed as before; and thus we may in all cases obtain an equation for determining all the values of u_0.

Rule for detecting the infinite values of u_0 immediately.

If in general we divide out x^m and the resulting equation for determining u_0 is only of the $n-m^{\text{th}}$ order, then m of the values of u_0 are infinite, as is easily seen. By remembering this we may detect the infinite values of u_0 immediately.

Example 1.

†† 159. Let the given equation be (see 153)

$$x^4 + 3a^2x^2 - 4a^2xy - a^2y^2 = 0,$$

here putting $y = ux$, dividing out x^2 and putting $x = 0$, we have

$$3a^2 - 4a^2u_0 - a^2u_0^2 = 0, \quad \text{or } u_0^2 - 4u_0 - 3 = 0,$$

which gives us the same values of $\frac{dy}{dx}$ that we obtained before.

Example 2.

Let the given equation be (see 153)

$$x^4 + ay^3 - 2axy^2 - 3ax^2y = 0,$$

here putting $y=ux$, dividing out x^3 and putting $x = 0$, we find

$$au_0^3 - 2au_0^2 - 3au_0 = 0,$$

$$\text{or } u_0^3 - 2u_0^2 - 3u_0 = 0,$$

which gives us the same values we obtained before of $\frac{dy}{dx}$.

Example 3.

Let the given equation be

$$y^5 + ax^4 - b^2xy^2 = 0,$$

in this case we find, putting $y = ux$ and dividing out x^3,

$$u_0^2 = 0,$$

hence, and therefore the three values of u_0 are 0, 0, and ∞.

How to proceed when the values of x and y which make $\frac{dy}{dx}$

†† 160. We have hitherto supposed that the values of x and y which make $\frac{dy}{dx}$ assume the form $\frac{0}{0}$ are $x = 0$ and $y = 0$: but let us now suppose them to be $x = a$ and $y = b$,

then putting $x = a + x_{,}$ and $y = b + y_{,}$ in the given equation $\frac{dy_{,}}{dx_{,}}$ will be the same thing as $\frac{dy}{dx}$, and $x_{,} = 0$ $y_{,} = 0$ will be the values of $x_{,}$ and $y_{,}$ which make $\frac{dy_{,}}{dx_{,}}$ assume the form $\frac{0}{0}$; we may therefore proceed as before. If the given equation be

assume the form $\frac{0}{0}$ are not zero.

$$U = 0,$$

and if we denote by $A, B, C, D, E \ldots$ &c. the values of $d_x U$, $d_y U$, $d_x^2 U$, $2 d_x d_y U$, $d_y^2 U$, &c. when x is put $= a$ and $\therefore\ y = b$; then, by 146, the result of substituting $a + x_{,}$ for x and $b + y_{,}$ for b will be

$$A x_{,} + B y_{,} + C x_{,}^2 + D x_{,} y_{,} + E y_{,}^2 + \text{\&c.} \ldots = 0;$$

if therefore A and B be not both zero, we have

$$A + B u_0 = 0,$$

and if A and B be both zero we have,

$$C + D u_0 + E u_0^2 = 0,$$

....

and so on...........

and we may find u_0 therefore just as before.

Let the given equation be (see 156) Example.

$$U = a y^2 - 2 a^2 y - 2 x^3 + 3 a x^2 = 0,$$

from which we wish to find $\frac{dy}{dx}$ when $x = a$, and $\therefore\ y = b$,

$$\text{then } d_x U = -6x^2 + 6ax = 0 \text{ when } x = a,$$
$$d_y U = 2ay - 2a^2 = 0 \text{ when } y = a,$$
$$d_x^2 U = -12x + 6a = -6a \text{ when } x = a,$$
$$d_x d_y U = 0,$$
$$d_y^2 U = 2a.$$

Hence $A = 0$, $B = 0$, $C = -6a$, $D = 0$, $E = 2a$, and therefore we have

$$-3 + u_0^2 = 0, \text{ and } \therefore\ u_0 = \pm\sqrt{3} \text{ as before.}$$

CHAPTER XI.

DETERMINATION OF THE MAXIMA AND MINIMA VALUES OF FUNCTIONS

Maxima and Minima values defined.

161. THE Differential Calculus may be applied with great success to determine the maxima and minima values of a function, i.e. those particular values which are either greater or less than any of the neighbouring values.

If $f(x)$ increases when x approaches a certain value a (supposing x to increase continually), and diminishes when x passes the value a, then $f(a)$ must be greater than any value of $f(x)$ which is either a little less or a little greater than a, and is therefore called a maximum value of $f(x)$. And if $f(x)$ diminishes when x approaches a and increases when x passes a, $f(a)$ must be less than any value of $f(x)$ which is either a little less or a little greater than a, and is therefore called a minimum value of $f(x)$.

How these values of $f(x)$ may be found by means of $f'(x)$.

162. Now by Lemma XVIII. $f(x)$ is increasing or diminishing according as $f'(x)$ is positive or negative: therefore if $f(a)$ be a maximum, $f'(x)$ must be positive for all values of x a little less than a, and negative for all values of x a little greater than a; i.e. $f'(x)$ must change its sign from $+$ to $-$ when x passes through the value a: and, conversely, if $f'(x)$ changes its sign from $+$ to $-$ when x passes through the value a, $f(a)$ is a maximum.

In like manner if $f(a)$ be a minimum, $f'(x)$ must change its sign from $-$ to $+$ when x passes through the value a; and conversely if $f'(x)$ does so change its sign, $f(a)$ is a minimum.

Hence by Lemma XVII if $f(a)$ be a maximum or minimum, $f'(a)$ must be zero or infinity: but of course, since $f'(a)$ may be zero or infinity without $f(x)$ changing its sign, it does

not follow that $f(a)$ must be a maximum or minimum whenever $f'(a)$ is zero or infinity.

163. *Hence to determine the maxima and minima of* $f(x)$, *we must determine what values of* x *make* $f'(x)$ *zero or infinity, and try whether* $f'(x)$ *changes its sign when* x *passes through each of these values; those values which give a change from* + *to* − *make* $f(x)$ *a maximum; those which give a change from* − *to* + *make* $f(x)$ *a minimum; and those which do not give a change must be rejected.* Rule for determining maxima and minima.

164. In applying this method to any example, we may suppress any factor of $f'(x)$ which we are sure is always positive, or may introduce any such factor; since we are not concerned with the actual magnitude of $f'(x)$ but only its sign. This will often considerably simplify our operations, as will appear. Factors of $f'(x)$ may be suppressed under certain circumstances

If therefore we find $f'(x)$ in the form $\phi(x) \, . \, \psi(x)$, and if $\phi(x)$ be always a positive quantity, then we may put $f'(x) = \psi(x)$ simply.

If $\phi(x)$ be always a negative quantity, then $f'(x)$ will have the same sign as $-\psi(x)$, and therefore we may put $f'(x) = -\psi(x)$.

165. If it be our object simply to examine whether $f'(x)$ changes its sign, and how, when x passes through a certain value a, then if $\phi(a)$ be neither zero nor infinity, we may suppose $f'(x) = +\psi(x)$ if $\phi(a)$ be positive, and $f'(x) = -\psi(x)$ if $\phi(a)$ be negative.

For it is clear that if $\phi(a)$ be neither zero nor infinity, then for all values of x near a $\phi(x)$ has the same sign as $\phi(a)$, and therefore $\phi(x) \, . \, \psi(x)$ or $f'(x)$ the same sign as $\phi(a)$ $\psi(x)$, i.e. $f'(x)$ has the same sign as $+\psi(x)$ or $-\psi(x)$ according as the sign of $\phi(a)$ is + or − for all values of x near a, and therefore in examining whether $f'(x)$ changes its sign, and how, when x passes through a, we may put $f'(x) = +\psi(x)$ or $-\psi(x)$ according as $\phi(a)$ is positive or negative.

We may therefore suppress all factors of $f'(x)$ which do not become 0 or ∞ when $x = a$, retaining only the signs they have when $x = a$. This is a very important simplification.

$\phi(u)$ is a maximum or minimum whenever u is, if $\phi(u)$ always increases when u increases if not the reverse is the case.

166. If $\phi(u)$ always increases when u increases, it is a maximum or minimum whenever u is a maximum or minimum respectively; for $\frac{d\phi(u)}{dx} = \phi'(u)\frac{du}{dx}$: and since $\phi(u)$ always increases when u increases, $\phi'(u)$ is always positive, and therefore $\frac{d\phi(u)}{dx}$ has always the same sign as $\frac{du}{dx}$, and $\therefore$ $\phi(u)$ and u become maxima or minima at the same time. If $\phi(u)$ diminishes when x increases the reverse is the case. We often find this consideration of use in practice, inasmuch as it may in many cases be much easier to find the maxima and minima of $\phi(u)$ than of u.

We may determine whether and how $f'(x)$ changes its sign when x passes through a, by the consideration of the second or higher differential coefficient.

167. If $f'(a) = 0$, then by Lemma XX, Cor. 4, $f'(x)$ has in general the same sign as $f^2(a)(x - a)$ for all values of x taken sufficiently near a: therefore if $f^2(a)$ be positive $f'(x)$ changes its sign from $-$ to $+$ when x passes through the value a, which indicates a ~~maximum~~ minimum: and if $f^2(a)$ be negative the change is from $+$ to $-$, which indicates a maximum. If however $f^2(a) = 0$, let $f^n(a)$ be the first differential coefficient of $f(x)$, which does not vanish when $x = a$; then $f'(x)$, (by Lemma XX, Cor. 4), has the same sign as $f^n(a)(x - a)^{n-1}$ for all values of x sufficiently near a. If, therefore, n be odd, $f'(x)$ does not change its sign when x passes through the value n; but if n be even it does, and the change is from $-$ to $+$ or from $+$ to $-$ according as $f^n(a)$ is positive or negative. Of course we here suppose that none of the differential coefficients are infinite.

Simple inspection is often the best way and sometimes the only way of making out whether $f'(x)$ changes its sign and how.

These considerations will enable us to determine whether $f'(x)$ changes its sign when x passes through the value a, and if so, whether the change is from $-$ to $+$ or from $+$ to $-$. But this is often more easily seen by simple inspection; and indeed when $f'(a)$ or any of the higher differential coefficients are infinite, which often occurs, simple inspection is the only method we can resort to.

By suppressing factors of $f'(x)$ which do not change their sign (in the manner shewn to be allowable in 164, 165,) before we differentiate $f'(x)$, we may often find the sign of $f^2(a)$, or more properly, of that quantity which answers the same purpose as $f^2(a)$, with great facility, *and avoid the necessity of being obliged to proceed to higher differential coefficients.*

168. If $y = b$ when $x = a$, and we can obtain a perfect developement in the form

When we obtain $y-b$ in the form $A(x-a)^m + R$, we may immediately see whether $x = a$ makes y a maximum or minimum.

$$y - b = A(x-a)^m + R = A(x-a)^m \left\{1 + \frac{R}{(x-a)^m}\right\},$$

then by taking x near enough to a, we may make $\frac{R}{(x-a)^m}$ as small as we please, and then $y - b$ will have the same sign as $A(x-a)^m$. Therefore if m be an odd number or a fraction with an odd numerator and an odd denominator and A positive, y is $< b$ when x is $< a$, and $> b$ when x is $> a$, and *vice versâ* if A be negative: in this case therefore b is not a maximum nor minimum value of y. But if m be an even number or a fraction with an even numerator and odd denominator and A positive, then y is $> b$ for all values of x near a, and therefore b is a minimum value of y: and if A be negative then y is less than b for all values of x near a, and therefore b is a maximum value of y. If m be a fraction in its lowest terms with an even denominator, then y is impossible for all values of x less than a $\{$or for all values of x greater than a if $y - b = A(a-x)^m + R\}$, in which case we cannot call b a maximum or minimum value of y.

169. The following examples will shew the advantages of the method of finding maxima and minima here recommended.

Example 1.

Let $$f(x) = x^5 (x-a)^3,$$

then $$f'(x) = x^4 (x-a)^2 (8x - 5a)$$

$$= 8x - 5a,$$

suppressing the factor $x^4 (x-a)^2$ which is always positive see (164).

Now $8x - 5a = 0$ when $x = \frac{5a}{8}$, and changes its sign

* When x is less or greater than $\frac{5a}{8}$.

from $-$ to $+$ * when x passes through or exceeds * the value $\frac{5a}{8}$: hence $x = \frac{5a}{8}$ gives a minimum value of $f(x)$, viz.

$$\left(\frac{5a}{8}\right)^5 \left(\frac{5a}{8} - a\right)^3 = -5^5 3^3 \left(\frac{a}{8}\right)^8.$$

Or thus $\quad f'(x) = 8x - 5a;$

$$\therefore f^2(x) = 8; \quad \therefore f^2\left(\frac{5a}{8}\right) = \text{positive},$$

which indicates a minimum by (167).

Example 2. Let $\quad f(x) = \frac{x^3}{(x-a)^5},$

then $f'(x) = \frac{x^2}{(x-a)^6}(-2x - 3a)$

$$= -(2x + 3a), \text{ suppressing the factor } \frac{x^2}{(x-a)^6}.$$

Now $-(2x + 3a)$ is zero when $x = -\frac{3a}{2}$, and changes its sign from $+$ to $-$ when x passes through that value: hence $x = -\frac{3a}{2}$ gives a maximum value of $f(x)$.

Or thus, $\quad f'(x) = -(2x + 3a)$

$$f^2(x) = -2; \quad \therefore f\left(-\frac{3a}{2}\right) = \text{negative},$$

which indicates a maximum.

Example 3. Let $\quad f(x) = \frac{\sin mx}{\sin x}$, m being an integer,

then $\quad f'(x) = \frac{m \cos mx \sin x - \sin mx \cos x}{\sin^2 x}$

$$= m \cos mx \sin x - \sin mx \cos x \text{ ...... (1)},$$

N.B. When the second Diff. Coef is positive, then fx is a minimum. When it is negative, then fx is a maximum

suppressing the factor $\frac{1}{\sin^2 x}$,

$$= \cos m x \cos x \,(m \tan x - \tan m x).$$

Now it is clear from geometrical considerations that there is some value of mx between 0 and $\frac{\pi}{2}$, which makes $\tan mx = m \tan x$: let this value be ma; then $f'(a) = 0$. Also differentiating (1),

$$f^2(x) = -(m^2 - 1) \sin mx \sin x,$$

which is negative when $x = a$, since m^2 is > 1, and ma and therefore a between 0 and $\frac{\pi}{2}$. Hence the root of the equation $\tan mx - m \tan x = 0$ which multiplied by m lies between 0 and $\frac{\pi}{2}$ makes $\frac{\sin mx}{\sin x}$ a maximum.

Let $f(x) = \left(\frac{\sin mx}{\sin x}\right)^2$, Example 4.

then $f'(x) = 2 \frac{\sin mx}{\sin^3 x} (m \cos mx \sin x - \sin mx \cos x)$

$$= \sin mx \sin x \,(m \cos mx \sin x - \sin mx \cos x),$$

multiplying by the factor $\frac{\sin^4 x}{2}$.

Now $x = a$ (see last example) makes $f'(x) = 0$; also $\sin mx \sin x$ is positive when $x = a$, and may therefore be suppressed so far as this value of x is concerned, therefore $f^2(x)$ has the same value as before and is therefore negative when $x = a$; which indicates a maximum.

Again, $x = \frac{\pi}{m}$ makes $\sin mx = 0$, and the product of the other factors of $f'(x)$ negative; therefore by (165) so far as this value of x is concerned we may put

$$f'(x) = -\sin mx;$$

$$\therefore\ f^2(x) = -m\cos mx = \text{positive when } x = \frac{\pi}{m},$$

hence $x = \dfrac{\pi}{m}$ gives a minimum.

Example 5. Let $$f(x) = (x-a)^{\frac{2}{3}},$$

$$\text{then}\quad f'(x) = \tfrac{2}{3}(x-a)^{-\frac{1}{3}}$$

$$= x - a,$$

multiplying by the factor $\frac{3}{2}(x-a)^{\frac{4}{3}}$ which is always positive;

$$\therefore\ f^2(x) = 1,$$

hence $f'(a) = 0$, and $f^2(a) = $ positive; and therefore $x = a$ gives a minimum.

(For more Examples see Appendix K.)

Maxima and minima of functions of several variables.

†† 170. Let $f(xy)$ be a function of two independant variables x and y, and let $f(x_1y)$ be a maximum value of $f(xy)$, determined on the supposition that y is constant and x alone variable, x_1 of course being some function of y; and again let $f(ab)$ be a maximum value of $f(x_1y)$ determined on the supposition that y is variable, a being the value of x_1 when y becomes b: then $f(ab)$ is a maximum value of $f(xy)$ when x and y are both supposed to vary in any manner.

For supposing x to have any value near x_1, $f(x_1y)$ is $> f(xy)$; and supposing y to have any value near b, and therefore x_1 some value near a, $f(ab)$ is $> f(x_1y)$, and therefore *à fortiori* $f(ab)$ is $> f(xy)$. Hence if xy have any values near a and b, $f(ab)$ is $> f(xy)$, and therefore $f(ab)$ is a maximum value of $f(xy)$, supposing x and y both to vary in any manner.

Partial and total maxima and minima.

We may call $f(ab)$ a *total* maximum in contradistinction to $f(x_1y)$, which is only a *partial* maximum determined on the supposition that only x varies.

In the same manner if $f(x_1y)$ be a *partial* minimum value of $f(xy)$, y being considered constant, and $f(ab)$ a

minimum value of $f(x_1 y)$, y being considered variable; then $f(ab)$ is a *total* minimum value of $f(xy)$.

Hence a maximum value of a partial maximum is a total maximum, and a minimum of a partial minimum is a total minimum.

Conversely, if $f(ab)$ be a total maximum, it must be the maximum of a partial maximum; for $f(ab)$ must be $> f(xy)$ for all values of x and y near a and b; therefore $f(ab)$ must be $> f(x_1 y)$, y being supposed to have any value near b, and therefore x some value near a; therefore $f(ab)$ is a maximum value of $f(x_1 y)$, i.e. it is a maximum of a partial maximum.

And similarly, if $f(ab)$ be a total minimum, it must be a minimum of a partial minimum.

Hence by finding the maxima of the partial maxima of $f(xy)$, and the minima of the partial minima, we find *all* the total maxima and minima values of $f(xy)$.

And thus, by the methods already given of finding maxima and minima of functions of one variable, we may obtain the total maxima and minima of functions of two independant variables.

Let $f(xy) = x^4 + y^4 - 4axy^2$, Example.

then to find the partial maxima and minima of $f(xy)$ on the supposition that x is variable, we have

$$d_x f(xy) = 4x^3 - 4ay^2,$$

which $= 0$ when $x = (ay^2)^{\frac{1}{3}}$, and evidently changes its sign from $-$ to $+$ when x passes through this value; which indicates a minimum. Therefore substituting $x = (ay^2)^{\frac{1}{3}}$ in $f(xy)$ we obtain the partial minimum

$$y^4 - 3a^{\frac{4}{3}} y^{\frac{8}{3}} = \psi(y) \text{ suppose.}$$

Then to find the minimum of this, we have

$$\psi'(y) = 4y^{\frac{5}{3}} (y^{\frac{4}{3}} - 2a^{\frac{4}{3}}),$$

which $= 0$ when $y = 0$ or $2^{\frac{3}{4}} a$: when y passes through the

former value $\psi'(y)$ changes its sign from + to −, and through the latter from − to +: the latter therefore makes $\psi'(y)$ a minimum.

Hence $f(xy)$ has a total minimum value obtained by putting $x = (ay^2)^{\frac{1}{3}}$ and $y = 2^{\frac{3}{4}}a$; i.e. $x = 2^{\frac{1}{2}}a$ and $y = 2^{\frac{3}{4}}a$.

(For more Examples see Appendix K.)

The equations $d_x f(xy) = 0$ or ∞, $d_y f(xy) = 0$ or ∞, give us those values of x and y, which may make $f(xy)$ a total maximum or minimum.

†† 171. Since $f(x_1 y)$ is a maximum or minimum on the supposition that y is constant, it is clear that $d_x f(xy)$ must become 0 or ∞ when x_1 is put for x, and this being true whatever value is assigned to y, it must be true when $y = b$ and therefore $x = a$. Hence if $f(ab)$ be a total maximum or minimum, we must have

$$d_x f(xy) = 0 \text{ or } \infty \;\ldots\; (1)$$

when a and b are put for x and y.

And in exactly the same way we may shew, by treating y as we have done x, and x as we have done y, that we must have

$$d_y f(xy) = 0 \text{ or } \infty \;\ldots\; (2)$$

when a and b are put for x and y.

a and b are therefore values of x and y got from the equations (1) and (2) taken together; and these equations therefore give us all the values of x and y which *may* make $f(xy)$ a total maximum or minimum.

The method of distinguishing maxima and minima of functions of two variables given by Lagrange is often troublesome; and since it does not include the cases where any of the partial differential coefficients of $f(xy)$ become infinite, or where those of the second order vanish, it must be considered as very incomplete.

How to find maximum and minimum values of y, when

†† 172. When y and x are connected by an equation, and we wish to find what values of x make y a maximum or minimum, we have only to differentiate the equation to find

$\dfrac{dy}{dx}$ and proceed as before. The following example will explain the process. an equation is given between x and y.

Given $y^4 - 4a^2xy + x^4 = 0 \ldots (1)$ to find what values of x make y a maximum or minimum.

We have, differentiating

$$(y^3 - a^2x)\frac{dy}{dx} + x^3 - a^2y = 0,$$

$$\text{and } \therefore \frac{dy}{dx} = \frac{x^3 - a^2y}{a^2x - y^3}.$$

$y = \dfrac{x^3}{a^2}$ makes this zero, (at least if x be not zero, for then $\dfrac{dy}{dx} = \dfrac{0}{0}$, a case which we shall consider presently); now if $y = \dfrac{x^3}{a^2}$ we have in virtue of (1)

$$\frac{x^{12}}{a^8} - 4a^2\frac{x^4}{a^2} + x^4 = 0 \quad \text{or} \quad x^8 - 3a^8 = 0, \quad \text{dividing out } x^4;$$

$$\text{or } x = 3^{\frac{1}{8}}a, \quad \text{and} \therefore y = 3^{\frac{3}{8}}a;$$

these values put in the denominator of $\dfrac{dy}{dx}$ make it negative; therefore by (165) we may put

$$\frac{dy}{dx} = a^2y - x^3;$$

$$\therefore \frac{d^2y}{dx^2} = a^2\frac{dy}{dx} - 3x^2$$

$$= \text{negative, when we put } x = 3^{\frac{1}{8}}a \text{ and } \therefore \frac{dy}{dx} = 0;$$

$$\therefore x = 3^{\frac{1}{8}}a \text{ makes } y \text{ a maximum.}$$

Next, as to the value $x = 0$ (and $\therefore y = 0$) we find, putting $y = ux$ as in 158 and dividing out x^2, that $u_0 = 0$, and $\therefore \dfrac{dy}{dx} = 0$

when $x=0$, therefore $x=0$ *may* give a maximum or minimum of y. To determine whether it does, put $y=ux^m$ as in 145, and we have

$$u^4x^{4m}-4a^2ux^{m+1}+x^4=0;$$

$4m=m+1$ gives $m=\frac{1}{3}$, and $4m$ or $m+1<4$; $\therefore$ we may assume $m=\frac{1}{3}$, and this gives $u^4-4a^2u=0$ when $x=0$; and $\therefore\ u=(4a^2)^{\frac{1}{3}}$, we therefore have

$$y=(4a^2)^{\frac{1}{3}}x^{\frac{1}{3}}+R.$$

Again, $4m=4$ makes $4m$ or $4>m+1$, and must therefore be rejected.

And again, $m+1=4$ makes $m=3$, and $\therefore\ m+1$ or $4<4m$; which gives us $u=\dfrac{1}{4a^2}$ when $x=0$, and

$$\therefore\ y=\frac{x^3}{4a^2}+R.$$

Now by (168), both these expressions for y shew that $y=0$ is neither a maximum nor a minimum value of y.

It appears from what has been just explained, that when we put $y=ux^m$ $\{$or $y-b=u(x-a)^m$ if $x=a$ and $y=b$ be the values we are concerned with$\}$, then we may immediately reject any value of m which is not an even number, or a fraction with an even numerator and odd denominator: bearing this in mind we may very readily obtain the maxima or minima values of y.

Lastly, $x=\dfrac{y^3}{a^2}$ makes $\dfrac{dy}{dx}=\infty$, and in virtue of (1) we have $y=3^{\frac{1}{4}}a$, $x=3^{\frac{3}{4}}a$.

Now assume $x=3^{\frac{3}{4}}a+z$, and $y=3^{\frac{1}{4}}a+uz^m$, and (1) becomes

$$(3^{\frac{1}{4}}a+uz^m)^4-4a^2(3^{\frac{1}{4}}a+uz^m)(3^{\frac{3}{4}}a+z)+(3^{\frac{3}{4}}a+z)^4=0,$$

which, bearing in mind the remark made in Appendix I respecting the rejection of certain terms, may be written

$$6 \,.\, 3^{\frac{2}{3}} a^2 u^2 z^{2m} - 4a^2 u z^{m+1} + 8 \,.\, 3^{\frac{1}{3}} a z = 0.$$

Here $2m = m + 1$ gives $m = 1$: $2m = 1$ gives $m = \frac{1}{2}$: and $m + 1 = 1$ gives $m = 0$. All these are to be rejected, since they are not even numbers or fractions with even numerators and odd denominators.

Hence it appears, that $x = 3^{\frac{1}{3}} a$ makes y a maximum, and neither $x = 0$ nor $x = 3^{\frac{2}{3}} a$ make y a maximum or minimum.

CHAPTER XII.

TANGENTS AND NORMALS TO CURVES. THE CURVATURE OF CURVES. THE EVOLUTE.

THE Differential Calculus is of great use in various parts of analytical geometry. We have already seen in (26) that it puts us in possession of a general method of drawing tangents to curves. We now proceed to shew that it admits of, not only this, but many other important applications in analytical geometry.

Proposition. To find the equation to the tangent at any point of a curve.

173. If a right line SPQ (fig. 6) be drawn passing through any two points of a curve, and if SPT be its limiting position when Q approaches P, i. e. if the angle made by PT and PQ may be diminished *ad libitum* by sufficiently diminishing the arc QP; then PT is said to be a tangent to the curve at the point P. Let $x\,y$ be the co-ordinates of P, $x'y'$ these of Q, $\angle PTX = \psi$, then $\tan\psi$ is the limiting value of $\tan PRX$ when Q approaches P, i. e. of $\dfrac{y'-y}{x'-x}$ when x' approaches x, which, by definition of a differential coefficient, is $\dfrac{dy}{dx}$; hence

$$\tan\psi = \frac{dy}{dx}.$$

Hence if $x_,\,y_,$ be the co-ordinates of any point on the tangent PT, since it is a line passing through the point P and making an angle $\tan^{-1}\dfrac{dy}{dx}$ with the axis of x, we have for its equation

$$y_, - y = \frac{dy}{dx}(x_, - x).$$

174. The normal at the point P is the line PG (fig. 8) which passes through P at right angles to the tangent PT: its equation is therefore

Cor. The equation to the Normal.

$$y_{,} - y = -\frac{dx}{dy}(x_{,} - x).$$

175. If we take s to represent the arc BP (fig. 6), s' the arc BQ, and c the cord PQ; then

Proposition. $ds^2 = dx^2 + dy^2$, s being the arc of the curve.

$$\frac{s' - s}{x' - x} = \frac{s' - s}{c} \cdot \frac{c}{x' - x}$$

$$= \frac{s' - s}{c} \sqrt{1 + \left(\frac{y' - y}{x' - x}\right)^2}, \text{ since } c^2 = (x' - x)^2 + (y' - y)^2.$$

Now when x' approaches x the limiting value of $\frac{s' - s}{x' - x}$ is $\frac{ds}{dx}$, that of $\frac{s' - s}{c}$ is 1 by Lemma IX, and that of $\frac{y' - y}{x' - x}$ is $\frac{dy}{dx}$, hence by Lemma VIII, we have

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}, \text{ or } ds^2 = dx^2 + dy^2.$$

176. We have $\cos PRX = \frac{x' - x}{c} = \frac{x' - x}{s' - s} \frac{s' - s}{c}$, and when x' approaches x the limiting value of PRX is ψ, that of $\frac{x' - x}{s' - s}$ is $\frac{dx}{ds}$, and that of $\frac{s' - s}{c}$ is 1; therefore

Cor. $dx = ds \cos\psi$, $dy = ds \sin\psi$.

$$\cos\psi = \frac{dx}{ds}, \text{ or } dx = ds \cos\psi.$$

Similarly $\sin PRX = \frac{y' - y}{c} = \frac{y' - y}{s' - s} \cdot \frac{s' - s}{c}$;

$$\therefore \sin\psi = \frac{dy}{ds}; \quad \therefore dy = ds \sin\psi.$$

These formulæ may be very simply obtained by the method of Leibnitz.

177. If it be allowable to argue according to the method of Leibnitz, spoken of in (84), we may suppose Q taken so near P that the arc QP may be regarded as a right line coinciding with the tangent at P, and PO, OQ, PQ taken respectively equal to dx, dy, ds; these differentials being regarded as infinitely small: then we have immediately

$$dx = ds\cos\psi, \quad dy = ds\sin\psi;$$

$$\text{and therefore } dx^2 + dy^2 = ds^2, \text{ and } \frac{dy}{dx} = \tan\psi.$$

Which formulæ are all correct results in consequence of a compensation of errors such as that shewn to exist in (85).

Though this method of arriving at these formulæ is not sufficiently exact for the purposes of elementary demonstration, it is useful in enabling us to remember and make out these formulæ.

Subtangent, subnormal and normal.

178. The line MT (fig. 8) is called the subtangent, MG the subnormal, PG the normal.

Hence

$$\text{subtangent} = y\cot\psi = \frac{y\,dx}{dy},$$

$$\text{subnormal} = y\tan\psi = \frac{y\,dy}{dx},$$

$$\text{normal} = y\sec\psi = \frac{y\,ds}{dx}.$$

(For examples of the use of these formulæ see Appendix I.)

[illegible]

179. Since $\psi = \tan^{-1}\dfrac{dy}{dx}$, we have

$$\frac{d\psi}{dx} = \frac{1}{1 + \left(\dfrac{dy}{dx}\right)^2}\cdot\frac{d^2y}{dx^2}.$$

Now $\dfrac{1}{1+\left(\dfrac{dy}{dx}\right)^2}$ is essentially a positive quantity; downwards, to the right or to the left.

hence $\dfrac{d\psi}{dx}$ has always the same sign as $\dfrac{d^2y}{dx^2}$.

Now if $\dfrac{d\psi}{dx}$ be positive, ψ increases when x increases, and therefore the concavity of the curve must be turned upwards, as in (fig. 9), but if $\dfrac{d\psi}{dx}$ be negative, ψ diminishes when x increases, and therefore the concavity of the curve is turned downwards, as in (fig. 10). Hence the concavity of the curve is turned upwards or downwards according as $\dfrac{d^2y}{dx^2}$ is positive or negative.

And in the same way it may be shewn that the concavity of the curve is turned to the right or to the left according as $\dfrac{d^2x}{dy^2}$ is positive or negative.

180. Hence if at any point $\dfrac{d^2y}{dx^2}$ changes its sign, the concavity of the curve will change its direction, if the change of sign is from + to − the concavity which was before turned upwards will begin to be turned downwards, as at the point P (fig. 11); and if the change be from − to + the reverse will be the case, as at the point P (fig. 12). Proposition. To determine points of contrary flexure.

A point where the concavity of the curve thus changes its direction is called a point of contrary flexure.

Hence, by Lemma XVII, $\dfrac{d^2y}{d^2x}$ must be 0 or ∞ at a point of contrary flexure; and therefore, to determine where such points are on the curve, we have only to find what values of x make $\dfrac{d^2y}{dx^2}$ 0 or ∞, and try whether x, in passing through

each of these values makes $\frac{d^2y}{dx^2}$ change its sign or not; those for which there is a change of sign give points of contrary flexure, and those for which there is not are to be rejected.

It is evident that ψ is a maximum or minimum at a point of contrary flexure.

(For Examples, see Appendix M.)

Proposition. To determine the degree of curvature at any point of a curve. Index of curvature.

181. $\frac{d\psi}{dx}$ measures the rate at which the tangent changes its position as we go along the axis of x, and of course its magnitude depends in part upon what position the axis of x occupies with reference to the curve. $\frac{d\psi}{ds}$ measures the rate at which the tangent changes its position as we go along the curve, and its magnitude does not at all depend upon the position of the co-ordinate axes, but solely upon the nature of the curve: the greater therefore the curvature at any point, the greater will be $\frac{d\psi}{ds}$, and *vice versâ*. $\frac{d\psi}{ds}$ may therefore be called the *Index of curvature* at any proposed point, since it indicates the rate at which the tangent changes its position as we go along the curve; i. e. the degree of curvature of the curve at each point.

Proposition respecting the quantity ρ.

182. *If* PS, QS, *fig.* 13, *be two normals to a curve at the points* P *and* Q, *and if* ρ *be the limiting value of* PS *when* Q *approaches* P, *and* c *the chord joining* PQ; *then* ρ *is the limiting value of* $\frac{c}{\angle S}$ *when* Q *approaches* P.

$$\text{For}\quad \frac{c}{S} = \frac{c}{\sin S}\cdot\frac{\sin S}{S}$$

$$= \frac{PS}{\sin PQS}\cdot\frac{\sin S}{S}.$$

Now by sufficiently diminishing PQ we may evidently make $\angle PQS$ differ from $\frac{\pi}{2}$ *ad libitum*; therefore when Q approaches P, 1 is the limiting value of $\sin PQS$; 1 is also the limiting

value of $\frac{\sin S}{S}$ by Lemma X, and we suppose ρ to be the limiting value of PS. Hence the limiting value of $\frac{c}{S}$ when Q approaches P is ρ. Q. E. D.

183. Let ψ and ψ' be the angles which the tangents at P and Q make with the axis of x, and s, s' the lengths of the arcs BP, BQ; then since $\psi' - \psi$ evidently $= \angle S$, we have

Cor. Index of curvature is equal to $\frac{1}{\rho}$.

$$\frac{\psi' - \psi}{s' - s} = \frac{\angle S}{c} \cdot \frac{c}{s' - s}.$$

Now when Q approaches P, the limiting value of $\frac{\psi' - \psi}{s' - s}$ is $\frac{d\psi}{ds}$, that of $\frac{\angle S}{c}$ is $\frac{1}{\rho}$ by what has just been proved, and that of $\frac{c}{s' - s}$ is 1 by Lemma IX: hence, we have

$$\frac{d\psi}{ds} = \frac{1}{\rho};$$

i.e. *the index of curvature* $= \frac{1}{\rho}$.

ρ is called the radius of curvature for reasons we shall hereafter explain.

184. We may find ρ in terms of x and y as follows:

Proposition. To find the value of ρ at any point of a given curve.

we have $\psi = \tan^{-1} \frac{dy}{dx}$; see (173).

$$\therefore\ d\psi = \frac{d\left(\frac{dy}{dx}\right)}{1 + \frac{dy}{dx^2}}$$

$$= \frac{dx^2}{ds^2}\, d\left(\frac{dy}{dx}\right), \text{ since } dx^2 + dy^2 = ds^2 \quad (175).$$

Hence, since $\frac{1}{\rho} = \frac{d\psi}{ds}$, we have

$$\frac{1}{\rho} = \frac{dx^2}{ds^3}\, d\left(\frac{dy}{dx}\right) = \frac{dx\,d^2y - d^2x\,dy}{ds^3} \text{......} (1).$$

If we suppose x the independant variable, this becomes

$$\frac{1}{\rho} = \frac{dx^2}{ds^3}\,\frac{d^2y}{dx} = \frac{dx\,d^2y}{ds^3} \text{.............} (2).$$

From the expressions (1) or (2) we may find $\frac{1}{\rho}$ when we know the curve.

Since $\frac{d\psi}{ds} = \frac{1}{\rho}$, it is clear from (179) that ρ is positive or negative according as the concavity of the curve at the point P is turned upward or downward.

(For examples, see Appendix N.)

Proposition. To determine the co-ordinates of the extremity of ρ.

185. Let P' (fig. 14) represent the limiting position of the point S in fig. 13, when Q approaches P; then PP' is the quantity we have denoted by ρ; let AL $(= \alpha)$, LP' $(= \beta)$, be the co-ordinates of P', PK parallel to LM. Then, since PP' evidently makes an angle $\frac{\pi}{2} - \psi$ with the line PK, we have

$$\left.\begin{aligned} \alpha &= x - \rho \sin \psi \\ \beta &= y + \rho \cos \psi \end{aligned}\right\} \text{.........} (1),$$

which equations determine the position of the point P' corresponding to the point xy of the curve.

Proposition. To find the curve traced out by P' when P moves along the curve BPQ.

186. It is evident that the point P' changes its position and traces out some curve when the point P is made to move along the curve BPQ; we may find the equation to the curve thus traced out as follows:

Since ρ, ψ, and y are all certain functions of x which we may find when the curve PQ is given, it is evident that we may eliminate x, y, ψ, ρ, from the equations (1), and so find a relation between α and β which is independent of x, and therefore holds for all positions of the point P. This relation is therefore the equation to the curve traced out by P'. This curve has some curious properties, as we proceed to shew.

187. Differentiating the equations (1) we find *Remarkable properties of this curve.*

$$d\alpha = dx - \rho\cos\psi d\psi - d\rho\sin\psi,$$
$$d\beta = dy - \rho\sin\psi d\psi + d\rho\cos\psi.$$

Now $\rho\cos\psi d\psi = \dfrac{ds}{d\psi}\dfrac{dx}{ds}d\psi$ (183, 176),

$$= dx,$$

and $\rho\sin\psi d\psi = dy$ similarly;

hence $d\alpha = -d\rho\sin\psi$,

$$d\beta = \quad d\rho\cos\psi.$$

from which equations we find

$$\frac{d\beta}{d\alpha} = -\cot\psi\ldots\ldots(2)$$

$$d\alpha^2 + d\beta^2 = d\rho^2\ldots\ldots\ldots\ldots(3).$$

188. Let $B'P'Q'$ (fig. 15) be the curve traced out by P'; then the equation to the tangent to this curve at the point P' is *First Property. The normal PP' touches the curve traced out by P' at the point P'.*

$$y_1 - \beta = \frac{d\beta}{d\alpha}(x_1 - \alpha),$$

or $y_1 - \beta = -\cot\psi\,.\,(x_1 - \alpha)$ by (2).

Now this is evidently the equation to the line PP' since that line passes through the point P', and makes an angle $\psi + \dfrac{\pi}{2}$ with the axis of x. Hence the normal PP' touches the curve traced out by P' at the point P' as we have represented in the figure.

189. Again, if σ denote the length of the arc $B'P'$ of the curve $B'P'Q'$ measured from some fixed point B', we have *Second Property. The curve BPQ is traced out by the end of a string unwound off the curve $B'P'Q'$.*

$$d\sigma = \sqrt{d\alpha^2 + d\beta^2} = \pm d\rho \text{ by (3)}:$$

since ρ evidently decreases when σ increases in consequence of the way we have measured σ, we must take the lower of these signs, and therefore we have

$$d\sigma + d\rho = 0;$$

$$\text{and} \quad \therefore \; d(B'P' + P'P) = 0,$$

i.e. the length $B'P' + P'P =$ some constant, C suppose.

Now let one end of a string whose length is C be fastened at B', and the other end to the point of a pencil; and the string being full stretched let it be wound on the curve $B'P'$ till it touches it at P', and therefore coincides with $P'P$ in direction: then since the length of the string is C, which $= B'P' + P'P$, the pencil point will be just at P; and this is true whatever point of the curve P' may be. Hence if we unwind the string off the curve $B'P'$, the pencil point will trace out the curve BPQ.

$B'P'Q$ is on this account called the evolute of BPQ.

It is on this account that the curve $B'P'Q'$ is called the evolute of the curve BPQ.

(For examples see Appendix O.)

CHAPTER XIII.

POLAR FORMULÆ, DIFFERENTIALS OF AREAS, SURFACES, AND VOLUMES.

We now proceed to investigate certain formulæ analogous to those of the last chapter, supposing the curve referred to polar co-ordinates.

190. Let APQ (fig. 16) be any curve referred to polar co-ordinates; $SP = r$, $PSA = \theta$; $SQ = r'$, $QSA = \theta'$, chord $PQ = c$, arc $AP = s$, arc $AP' = s'$: then Proposition. To find $\frac{ds}{d\theta}$.

$$\frac{s' - s}{\theta' - \theta} = \frac{s' - s}{c} \, \frac{c}{\theta' - \theta}.$$

Now
$$c^2 = r'^2 + r - 2rr' \cos(\theta' - \theta)$$
$$= (r' - r)^2 + 4rr' \sin^2 x,$$

putting for brevity $x = \dfrac{\theta' - \theta}{2}$, and $\cos \theta' - \theta = 1 - 2\sin^2 x$.

Hence
$$\frac{s' - s}{\theta' - \theta} = \frac{s' - s}{c} \sqrt{\left(\frac{r' - r}{\theta' - \theta}\right)^2 + rr' \left(\frac{\sin x}{x}\right)^2}.$$

Let θ' approach θ, and therefore x zero; then the limiting value of $\dfrac{s' - s}{\theta' - \theta}$ is $\dfrac{ds}{d\theta}$, that of $\dfrac{s' - s}{c}$ is 1, that of $\dfrac{\sin x}{x}$ is also 1, and that of r' is r: therefore we have

$$\frac{ds}{d\theta} = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}, \quad \text{or } ds = \sqrt{dr^2 + r^2 d\theta^2}.$$

Cor. This result deduced from the rectangular formula.

191. This result we may also prove as follows.

By (175) $ds = \sqrt{dx^2 + dy^2}$,

$$\text{but} \quad x = r\cos\theta, \quad y = r\sin\theta;$$

$$\therefore \; dx = dr\cos\theta - r\sin\theta\, d\theta,$$

$$dy = dr\sin\theta + r\cos\theta\, d\theta;$$

$$\therefore \; dx^2 + dy^2 = dr^2 + r^2 d\theta^2; \text{ and } \therefore \; ds = \sqrt{dr^2 + r^2 d\theta^2}.$$

Proposition. To find p the perpendicular on the tangent from S.

192. Draw SY perpendicular to the line QPT which passes through the points P and Q, and draw PO perpendicular to SQ, then by similar triangles

$$SY = SQ\,.\frac{OP}{PQ} = r'\frac{r\sin(\theta' - \theta)}{c}$$

$$= r'r\,.\frac{\sin(\theta' - \theta)}{\theta' - \theta}\,.\frac{\theta' - \theta}{s' - s}\,.\frac{s' - s}{c}.$$

Now let θ' approach θ; then the limiting position of the line QPT will be the tangent at P; and therefore if p be the perpendicular from S upon that tangent, the limiting value of SY will be p. Hence, by Lemmas IX and X, we have

$$p = r^2\frac{d\theta}{ds}.$$

Cor. This result deduced from rectangular formulæ.

193. This result we may also prove as follows.

The equation to the tangent is

$$y_1 - y = \frac{dy}{dx}(x_1 - x),$$

and the perpendicular p upon this line from the origin is

$$\frac{x\dfrac{dy}{dx} - y}{\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}}, \text{ or } \frac{xdy - ydx}{ds}, \text{ since } ds = \sqrt{dx^2 + dy^2};$$

hence, since $y = r\sin\theta$, $x = r\cos\theta$;

and therefore $dy = dr\sin\theta + r\cos\theta\,d\theta$,

$$dx = dr\cos\theta - r\cos\theta\,d\theta,$$

we have

$$xdy - ydx = r^2 d\theta,$$

and therefore $p = \dfrac{r^2 d\theta}{ds}$.

194. Sin $OQP = \dfrac{OP}{c} = \dfrac{r\sin(\theta' - \theta)}{c}$ Proposition. To find the sine and cosine of the angle under the tangent and radius vector, viz. ϕ.

$$= \frac{r\sin(\theta' - \theta)}{\theta' - \theta} \cdot \frac{\theta' - \theta}{s' - s} \cdot \frac{s' - s}{c}.$$

Now let θ' approach θ, then if ϕ be the angle which the tangent at P makes with the radius vector SP, the limiting value of OQP is ϕ. Hence we have

$$\sin\phi = \frac{rd\theta}{ds}.$$

Hence $$\cos\phi = \sqrt{1 - \left(\frac{rd\theta}{ds}\right)^2}$$

$$= \sqrt{\frac{ds^2 - r^2 d\theta^2}{ds^2}}$$

$$= \frac{dr}{ds} \text{ by (191).}$$

Hence $$\tan\phi = \frac{rd\theta}{dr}.$$

195. Since $p = r\sin\phi$, we have from this Cor. p hence found.

$$p = \frac{r^2 d\theta}{ds} \text{ as before.}$$

Proposition. To find $\tan\phi$ independantly of the preceding articles.

196. $$\tan OQP = \frac{OP}{OQ} = \frac{r\sin(\theta'-\theta)}{r'-r\cos(\theta'-\theta)}$$

Here put $\dfrac{\theta'-\theta}{2} = z$, and $\cos(\theta'-\theta) = 1 - 2\sin^2 z$,

$$\text{then } \tan OQP = \frac{r\sin(\theta'-\theta)}{r'-r+2r\sin^2 z}$$

$$= \frac{\dfrac{r\sin(\theta'-\theta)}{\theta'-\theta}}{\dfrac{r'-r}{\theta'-\theta} + r\dfrac{\sin z}{z}\sin z}.$$

Now let θ' approach θ, and therefore z zero; then the limiting values of $\dfrac{\sin(\theta'-\theta)}{\theta'-\theta}$, $\dfrac{r'-r}{\theta'-\theta}$, $\dfrac{\sin z}{z}$, and $\sin z$, are respectively 1, $\dfrac{dr}{d\theta}$, 1, and zero; hence we have

$$\tan\phi = \frac{r}{\dfrac{dr}{d\theta}} = \frac{rd\theta}{dr}.$$

Cor. To find the polar subtangent.

197. The line drawn from S perpendicular to the radius vector to meet the tangent is called the polar subtangent; it evidently is equal to

$$r\tan\phi, \text{ or } \frac{r^2 d\theta}{dr}.$$

These formulæ proved by Leibnitz' method very simply.

198. If it be allowable to argue according to the method of Leibnitz, spoken of in (84), we may, as in (177), suppose

$$QP = ds, \quad QO = dr, \quad OP = rd\theta, \quad \angle OQP = \phi:$$

and hence we get

$$ds = \sqrt{r^2 d\theta^2 + dr^2}, \quad \sin\phi = \frac{rd\theta}{ds}, \quad \tan\phi = \frac{rd\theta}{dr},$$

$$p = r\sin\phi = \frac{r^2 d\theta}{ds}, \text{ polar subtangent } = r\tan\phi = \frac{r^2 d\theta}{dr}.$$

This method of arriving at these formulæ will enable us to make out any of them very readily, in case we forget them.

(For Examples of the use of these formulæ see Appendix P.)

199. If ψ be the angle made by PT with SA, then we have

Proposition. To find ρ in terms of r and p.

$$\psi = \theta + \phi, \text{ and } \therefore d\psi = d\theta + d\phi.$$

Now $p = r\sin\phi$, and

$$\therefore dp = dr\sin\phi + r\cos\phi\, d\phi$$

$$= dr\frac{rd\theta}{ds} + \frac{rdr}{ds}d\phi$$

$$= \frac{rdr}{ds}(d\theta + d\phi) = \frac{rdr}{ds}d\psi$$

$$= \frac{rdr}{\rho}, \text{ since } \frac{d\psi}{ds} = \frac{1}{\rho} \quad (183);$$

$$\text{hence } \rho = \frac{rdr}{dp}.$$

200. We have as in (199)

Proposition. To find ρ in terms of r and θ.

$$\psi = \phi + \theta = \tan^{-1}\frac{rd\theta}{dr} + \theta,$$

hence differentiating, supposing $d\theta$ constant, we have

$$d\psi = d\theta\left(\frac{1 - \dfrac{rd^2r}{dr^2}}{1 + \dfrac{r^2d\theta^2}{dr^2}} + 1\right)$$

$$= d\theta\frac{2dr^2 - rd^2r + r^2d\theta^2}{ds^2}, \text{ since } ds^2 = dr^2 + r^2d\theta^2,$$

$$\text{hence } \rho = \frac{ds}{d\psi} = \frac{ds^3}{d\theta\,(2dr^2 - rd^2r + r^2d\theta^2)}$$

$$\text{or } \frac{ds^3}{d\theta\,(dr^2 + ds^2 - rd^2r)}.$$

Cor. The same result deduced from rectangular formulæ.

201. This result may also be arrived at thus.

We have by (186) $\rho = \dfrac{ds^3}{d^2y\,dx - d^2x\,dy}$.

$$\begin{aligned} \text{Now}\quad dy &= dr\sin\theta + r\cos d\theta, \\ d^2y &= d^2r\sin\theta + 2dr\cos\theta\,d\theta - r\sin\theta\,d\theta^2, \\ dx &= dr\cos\theta - r\sin\theta\,d\theta. \end{aligned}$$

Hence

$$\begin{aligned} d^2y\,dx = (drd^2r &- 3rdrd\theta^2)\sin\theta\cos\theta \\ &+ 2dr^2\cos^2\theta\,.\,d\theta - r\,(d^2r - rd\theta^2)\sin^2\theta\,.\,d\theta. \end{aligned}$$

Now $d^2x\,dy$ is evidently got from this by putting $\dfrac{\pi}{2} - \theta$ instead of θ; hence we evidently have, putting $\sin\theta$ for $\cos\theta$, $\cos\theta$ for $\sin\theta$, $-d\theta$ for $d\theta$, and subtracting

$$d^2y\,dx - d^2x\,dy = d\theta\,(2dr^2 - rd^2r + r^2d\theta^2),$$

$$\text{and } \therefore \rho = \frac{ds^3}{d\theta\,(2dr^2 - rd^2r + r^2d\theta^2)}.$$

Proposition. To find whether the concavity of the curve is turned to or from the pole.

202. If p increases when r increases, it is easy to see that the concavity of the curve is turned towards the pole, and if p diminishes when r increases, it is turned from the pole. Hence, by Lemma XVIII., the concavity of the curve is turned towards or from the pole according as $\dfrac{dp}{dr}$ is positive or negative.

(For Examples see Appendix P.)

The use of the following formulæ respecting the differentials of areas, surfaces, and volumes cannot be shewn without the assistance of the Integral Calculus.

The Integral Calculus is the inverse of the Differential Calculus, its object being to find the quantity from which a given differential is derived. When the student becomes acquainted with it, he will perceive the importance of these formulæ.

203. Let BPQ (fig. 17) be any curve, $AM = x$, $MP = y$, $AN = x'$, $NQ = y'$, the co-ordinates of any two points P and Q of it. Let A denote the area BPM and A' the area BQN; draw PO and QR parallel to MN to meet NQ and MP in O and R.

Proposition. To find the differential of an area.

Then area $MPQN$ lies between $MRQN$ and $MPON$,

$$\text{i.e. } A' - A \text{ lies between } y'(x' - x) \text{ and } y(x' - x);$$

$$\therefore \frac{A' - A}{x' - x} \text{ lies between } y' \text{ and } y.$$

Now $y' = y$ when $x' = x$, hence by Lemma VII. y is the limiting value of $\dfrac{A' - A}{x' - x}$ when x' approaches x, i.e.

$$\frac{dA}{dx} = y, \quad dA = y\,dx.$$

204. Let V and V' be the volumes generated by the revolution of the areas BPM and BQN (fig. 17) round AN, then

Proposition. To find the differential of a solid of revolution.

vol. gen. by $MPQN$ lies between vol. gen. by $MPON$ and vol. gen. by $MRQN$,

$$\text{i.e. } V' - V \text{ lies between } \pi y^2 (x' - x) \text{ and } \pi y'^2 (x' - x);$$

$$\therefore \frac{V' - V}{x' - x} \text{ lies between } \pi y^2 \text{ and } \pi y'^2;$$

$$\therefore \text{ as before } \frac{dV}{dx} = \pi y^2, \quad dV = \pi y^2 dx.$$

205. Let S and S' denote the surfaces generated by the revolution of the arcs BP and BQ about AN (fig. 17); let s denote the former arc and s' the latter; produce PO

Proposition. To find the differential

of a surface of revolution. to U and QR to T, so that PT and QU shall each be equal to the arc PQ. Then we may evidently assume that

surf. gen. by PQ lies between surf. gen. by PT and surf. gen. by QU,

i.e. $S' - S$ lies between $2\pi y(s'-s)$ and $2\pi y'(s'-s)$,

$$\therefore \frac{S'-S}{s'-s} \text{ lies between } 2\pi y \text{ and } 2\pi y'.$$

Hence as before $\dfrac{dS}{ds} = 2\pi y$, or $dS = 2\pi y\sqrt{dx^2+dy^2}$.

Proposition. To find the differential of a polar area.

206. Let APQ (fig. 18) be a curve referred to polar co-ordinates, $SP = r$, $SQ = r'$, $PSA = \theta$, $QSA = \theta'$: let A and A' be the areas ASP and ASQ; describe the circular arcs PO and QR round S as center. Then

area SPQ lies between area SPO and area SQR,

i.e. $A' - A$ lies between $\dfrac{r^2(\theta'-\theta)}{2}$ and $\dfrac{r'^2(\theta'-\theta)}{2}$;

$$\therefore \frac{A'-A}{\theta'-\theta} \text{ lies between } \frac{r^2}{2} \text{ and } \frac{r'^2}{2}.$$

Hence as before $\dfrac{dA}{d\theta} = \dfrac{r^2}{2}$.

Cor. The same result deduced from the rectangular formula.

207. This result may also be obtained as follows: draw PM perpendicular to SM, then $SM = x$ and $PM = y$ will be the rectangular co-ordinates of P, let B be the area APM, then by 203

$$dB = y\,dx,$$

$$\text{but } B = \text{area } SPM - \text{area } SPA$$

$$= \frac{xy}{2} - A;$$

$$\therefore \frac{x\,dy + y\,dx}{2} - dA = y\,dx;$$

$$\therefore dA = \frac{x\,dy - y\,dx}{2}.$$

Hence as in (193),

$$dA = \frac{r^2 d\theta}{2}.$$

208. The result

$$2dA = xdy - ydx$$ $2dA = xdy - ydx.$

is of importance.

CHAPTER XIV.

ASYMPTOTES.

An Asymptote, what.

209. It often happens, in the case of curves which have infinite branches, that when the point of contact is moved off to an infinite distance from the origin, the tangent remains at a finite distance, or, to speak more accurately, approaches a certain limiting position which is at a finite distance from the origin. In such a case the tangent or rather its limiting position is called an asymptote.

Proposition. To determine the equation to an asymptote of a curve.

210. The equation to an asymptote, when one exists, may be thus determined.

In the equation to the tangent, which may be written in the form

$$y_{,} = \frac{dy}{dx} x_{,} + y - \frac{dy}{dx} x,$$

$$\text{put } \frac{y}{x} = u, \quad \text{and therefore } \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = \frac{du}{dx},$$

and it evidently becomes

$$y_{,} = \left(\frac{y}{x} + x\frac{du}{dx}\right) x_{,} - x^2 \frac{du}{dx},$$

put moreover

$$x = \frac{1}{z}, \quad \text{and} \therefore dx = -\frac{dz}{z^2} = -x^2 dz,$$

and the equation reduces to the following form

$$y_{,} = \left(u - z\frac{du}{dz}\right) x_{,} + \frac{du}{dz}.$$

Now if the curve be of such a nature that when x approaches ∞, i. e. when z approaches zero, the limiting values

of u and $\dfrac{du}{dz}$ are finite quantities A and B; then the limiting form which the equation to the tangent assumes is

$$y_{\prime} = Ax_{\prime} + B,$$

which therefore is the equation to the asymptote.

211. It is very easy to find A and B in all cases from the equation to the curve, by putting $x = \dfrac{1}{z}$, and $y = \dfrac{u}{z}$ in that equation, and so finding a relation between u and z: then A being the limiting value of u when z approaches zero may be found by putting $z = 0$, and B, which is the limiting value of $\dfrac{du}{dz}$ when z approaches zero, may be found by differentiating the equation and putting $z = 0$. How to find A and B.

Let $x^3 + 3axy - y^3 = 0$ be the equation to the curve; then putting $x = \dfrac{1}{z}$, $y = \dfrac{u}{z}$ and multiplying by z^3 we find Example.

$$1 + 3auz - u^3 = 0,$$

and differentiating $3au + (3az - 3u^2)\dfrac{du}{dz} = 0$;

in these equations put $z=0$, and $\therefore u=A$, $\dfrac{du}{dz} = B$, and we find

$$1 - A^3 = 0, \quad 3aA - 3A^2B = 0;$$

$$\therefore A = 1, \quad B = A,$$

and the equation to the asymptote is therefore

$$y = x + a.$$

212. If A and B turn out to be impossible or infinite, then there is no asymptote corresponding to an infinite value of the abscissa: but there may be one corresponding to a finite value of x, for the point of contact may go off to an infinite distance from the origin for a finite value of x, y of course becoming infinite. Such an asymptote cannot be found by the There may be an asymptote corresponding to a finite value of x which must be found otherwise.

method just explained, since in that method we suppose x infinite; but it is easy to see that whenever the ordinate becomes infinite for a finite value of x, it becomes an asymptote, and therefore if a be a finite value of x which makes y infinite, the line whose equation is $x = a$ is an asymptote. Therefore all we have to do in a case of this kind is to put $y = \frac{1}{z}$ in the equation to the curve, and then make $z = 0$; and if this gives us a finite value of x, a suppose, $x = a$ is the equation to the asymptote.

Example. Let the equation to the curve be

$$y^2 x - x^3 - 2ay^2 = 0,$$

putting $y = \frac{u}{z}$, $x = \frac{1}{z}$ and multiplying by z^3 we find

$$u^2 - 1 - 2au^2 z = 0,$$

$$\text{and differentiating } (2u - 4auz)\frac{du}{dz} - 2au^2 = 0,$$

in these equations putting $z = 0$, and $\therefore u = A$, $\frac{du}{dx} = B$, we find

$$A^2 - 1 = 0, \quad B - aA = 0,$$

$$\therefore A = \pm 1, \quad B = \pm a,$$

hence there are two asymptotes corresponding to $x = \infty$ whose equations are

$$y = x + a, \text{ and } y = -x - a.$$

Moreover, putting $y = \frac{1}{z}$ in the original equation and multiplying by z^2, we find

$$x - x^3 z^2 - 2a = 0,$$

and therefore when $z = 0$, $x = 2a$, which is a finite value; hence there is another asymptote whose equation is

$$x = 2a.$$

When therefore our object is to find all the asymptotes of a curve, we must not forget to try whether there is one corresponding to a finite value of x.

213. We may consider asymptotes somewhat differently, and perhaps more simply, in the following manner. **Asymptotes considered somewhat differently.**

By putting $\frac{1}{z}$ for x, and $\frac{u}{z}$ for y in the equation to the curve, we find a relation between u and z; and if u and $\frac{du}{dz}$ have finite values A and B when $z = 0$, we may by Taylor's Theorem assume that

$$u = A + Bz + R;$$

where R is some quantity, such that zero is the limiting value of $\frac{R}{z}$ (i. e. of Rx) when z approaches zero, i. e. when x approaches infinity.

Therefore, restoring x and y, we may assume that

$$y = Ax + B + Rx.$$

Now let BP, fig. 19, represent the curve, $AM = x$, $MP = y$: also let $B'P'$ be the line whose equation is

$$y' = Ax + B, \text{ where } y' = MP',$$

$$\text{then } PP' = y - y' = Rx;$$

therefore, since zero is the limiting value of Rx when x approaches ∞, PP' may be diminished *ad libitum* by sufficiently increasing x. Therefore the line $B'P'$ continually approaches the curve BP as we go off to an infinite distance from the origin, but never actually meets it, though we may make PP' as small as we please. A line thus circumstanced is called an Asymptote, and we may find its equation just as before, since A and B are the same quantities as before*.

214. We may determine the asymptotes of a polar curve as follows: **Proposition.**

* See note, page 154.

To determine the asymptotes to a polar curve by means of what has been proved.

Let the equation to the curve be put in the form

$$r = \frac{1}{f(\theta)} \text{ .... (1)},$$

then, since $x = r\cos\theta$, $y = r\sin\theta$, we have, putting $\frac{1}{x}$ for x, and $\frac{u}{x}$ for y,

$$x = \frac{1}{r\cos\theta} = \frac{f(\theta)}{\cos\theta} \dots (1) \quad u = \tan\theta \dots (2);$$

$$\text{and } \therefore \frac{du}{dx} = \frac{d\tan\theta}{d\left\{\dfrac{f(\theta)}{\cos\theta}\right\}} = \frac{1}{f'(\theta)\cos\theta + f(\theta)\sin\theta} \dots (3).$$

Now let α be a value of θ which makes r infinite, and therefore

$$f(\theta) \left(\text{which } = \frac{1}{r}\right) = 0;$$

then $\theta = \alpha$ makes $x = 0$ by (1)*; and therefore if A and B be the values of u and $\frac{du}{dx}$ when $x = 0$, we have by (2) and (3),

$$A = \tan\alpha, \quad B = \frac{1}{f'(\alpha)\cos\alpha};$$

and therefore the rectangular equation to the asymptote required is

$$y = \tan\alpha \,.\, x + \frac{1}{f'(\alpha)\cos\alpha}.$$

$$\text{or} \quad x = -\frac{1}{f'\left(\dfrac{\pi}{2}\right)} \text{ if } \alpha = \frac{\pi}{2}; \text{ see note.}$$

* Except when $\alpha = \frac{\pi}{2}$, in which case x assumes the form $\frac{0}{0}$ when $\theta = \alpha$, and its limiting value when θ approaches α is $-f\left(\frac{\pi}{2}\right)$; $\therefore x = -\frac{1}{f\left(\frac{\pi}{2}\right)}$ makes r and therefore y infinite; therefore by 212 $x = -\frac{1}{f'\left(\frac{\pi}{2}\right)}$ is the equation to the asymptote.

215. We may also obtain this result as follows. **Cor. Same result obtained independantly.**

Let α be the angle which the asymptote makes with the prime radius; then it is evident that $\theta = \alpha$ ought to make r infinite, and therefore $f(\alpha)$ must be zero. Let the rectangular equation to the asymptote be

$$y = \tan\alpha\,(x + c) \;\ldots\ldots\ldots\ldots\; (2),$$

which, putting $r'\cos\theta$ $r'\sin\theta$ for x and y, becomes

$$r' = \frac{c\sin\alpha}{\sin(\theta - \alpha)};$$

$$\text{hence } \frac{r'}{r} = c\sin\alpha \,.\, \frac{f(\theta)}{\sin(\theta - \alpha)} = \frac{0}{0} \text{ when } \theta = \alpha.$$

Now the limiting value of $\frac{r'}{r}$ when θ approaches α ought to be unity, from the nature of an asymptote; but by the usual method of vanishing fractions this limiting value is $c\sin\alpha\, f'(\alpha)$.

Hence we have $c\sin\alpha f'(\alpha) = 1$;

$$\text{and } \therefore\ c\sin\alpha = \frac{1}{f'(\alpha)}.$$

The equation to the asymptote is therefore {substituting for c in (2)},

$$y = x\tan\alpha + \frac{1}{\cos\alpha f'(\alpha)}.$$

We have here obtained the rectangular equation to the asymptote, because it is easier to make use of it than the polar.

Let $r = \frac{a}{\theta}$ be the equation to the curve; here **Example 1.**

$$f(\theta) = \frac{\theta}{a};\ \text{and } \therefore\ \alpha = 0;\ \text{also } f'(\theta) = \frac{1}{a};\ \therefore\ f'(\alpha) = \frac{1}{a};$$

therefore the equation to the asymptote is $y = a$, which represents a line parallel to the axis of x at a distance a above it. See fig. 20.

Let $r = \frac{a(e^2 - 1)}{1 - e\cos\theta}$ (a hyperbola referred to focus), **Example 2.**

$$\text{here}\quad f(\theta)=\frac{1-e\cos\theta}{a(e^2-1)};$$

$$\text{and}\ \therefore\ \cos\alpha=\frac{1}{e},\quad \sin\alpha=\pm\sqrt{1-\frac{1}{e^2}}=\pm\frac{\sqrt{e^2-1}}{e},$$

$$\tan\alpha=\pm\sqrt{e^2-1}.$$

$$\text{Also}\quad f'(\theta)=\frac{e\sin\theta}{a(e^2-1)};\quad \therefore\ f'(\alpha)=\pm\frac{1}{a\sqrt{e^2-1}};$$

hence the equation to the asymptote is

$$y=\pm\sqrt{e^2-1}\,.\,x\pm ae\sqrt{e^2-1},$$

$$\text{or}\quad y=\pm\frac{b}{a}(x+ae),$$

which shews that there are two asymptotes making angles $\tan^{-1}\dfrac{b}{a}$, and $-\tan^{-1}\dfrac{b}{a}$ with the axis of x, and meeting it at a distance ae behind the origin.

Asymptotic circle, what.

216. It sometimes happens that r assumes a finite value, c suppose, when we put $\theta=\infty$ in the polar equation to a curve: it is easy to see that in such a case if we describe a circle round the pole with radius c, we may by continually increasing θ make the curve approach as near as we please to this circle without ever actually meeting it. Such a circle is called an asymptotic circle.

Example.

Let $r=\dfrac{a\theta}{\pi+\theta}$; then $\theta=\infty$ makes $r=a$; therefore by continually increasing θ we may diminish $r\sim a$ *ad libitum*, and therefore make the curve approach as near to the circle as we please without ever actually meeting it, since it requires an infinite number of revolutions of r to make $r=a$.

The circle in this case is an exterior asymptote, since r is evidently always less than a, (see fig. 21).

If $r=\dfrac{a\theta}{\theta-\pi}$ be the equation, r is always greater than a (at least when θ is taken large enough), and therefore the circle is an exterior asymptote.

CHAPTER XV.

ON THE METHOD OF TRACING THE GENERAL FORM OF A CURVE FROM ITS EQUATIONS.

It is often necessary to make out and trace the general form of a curve from its equation, without actually calculating its exact dimensions, or ascertaining the precise positions of its remarkable points: we now proceed to state how this may be done in most cases.

217. If we can find y in terms of x, it is easy in general to trace the general form of the curve by determining the values of x which make $y=0$ or ∞, the corresponding values of $\frac{dy}{dx}$, and the signs which y has between its zero and infinite values. We may do this in the following manner, viz. In one column write in order the values of x which make $y=0$ or ∞, and in addition to these, the value $x=\infty$: opposite to these values, in another column, write down the corresponding values of y, and between each two put the sign which y has between them (y will always have the same sign between each two of these values, since it can only change its sign in passing through 0 or ∞): and in a third column put the corresponding values of $\frac{dy}{dx}$. By means of such a table it will be easy, in most cases, to trace the general form of the curve, as the following example will shew. **How to trace a curve when we can find y in terms of x.**

218. To trace the general form of the curve whose equation is **Example.**

$$y=\frac{x^2}{a}\,\frac{x+a}{x-a};$$

here y is zero when $x=0$ or $-a$, and infinite when $x=a$: hence the values of x to be written down are $-a$, 0, a, ∞;

the corresponding values of y are 0, 0, ∞, ∞: and by differentiating, or rather by the method given in the note*, we shall find that the corresponding values of $\frac{dy}{dx}$ are $-\frac{1}{2}$, 0, ∞, ∞†; also when x is $<-a$ the sign of y is +, when

How to find $f'(a)$ very readily when $f(a) = 0$ or ∞.

* To find the value of the differential coefficient of $f(x)$ when $x=a$, supposing a to be a value of x which makes $f(x)=0$ or ∞, we have only to divide $f(x)$ by $x-a$ and then put $x=a$: for if $f(a)=0$, the limiting value of $\frac{f(x)}{x-a}$ when x approaches a is $f'(a)$ by Lemma XIX.; and if $f(a)=\infty$, $\frac{f(x)}{x-a}$ becomes ∞ when $x=a$, and thus gives the proper value of $f'(a)$, which by Lemma XXIII. we know to be ∞. *In either case therefore, if we divide* f(x) *by* x − a *and put* x = a, *the result will be the proper value of* f′(a).

Thus let $f(x)=\frac{x^2}{a}\frac{x+a}{x-a}$; then if we divide by $x+a$ and put $x=-a$ we obtain immediately $f'(-a)=-\frac{1}{2}$. If we divide by x and put $x=0$, we find $f'(0)=0$. If we divide by $x-a$ and put $x=a$, we find $f'(a)=\infty$.

To find y and $\frac{dy}{dx}$ when $x=\infty$.

† The values of y and $\frac{dy}{dx}$ when $x=\infty$ may be easily found thus.

Suppose y to be in the form of a fraction $\frac{u}{v}$, put u in the form $x^m(A+R)$ and v in the form $x^n(B+R')$, where R and R' are quantities which vanish when $x=\infty$;

$$\text{then} \quad y=x^{m-n}\frac{A+R}{B+R'};$$

and therefore, if $m=n$, $y=\frac{A}{B}$ when $x=\infty$; if m be $<n$, $y=0$ when $x=\infty$; and if m be $>n$, $y=\infty$ when $x=\infty$. It is always quite easy to put an algebraical function of x in the form $x^m(A+R)$ by examining which term contains the highest power of x, and making that power appear as a factor of the whole. Thus

$$x^3+3x^2+6=x^3\left(3+\frac{1}{x}+\frac{6}{x^3}\right)=x^3(3+R),$$

$$a^2x+ax\sqrt{x^2-a^2}+ax^2=x^2\left(a+a\sqrt{1-\frac{a^2}{x^2}}+\frac{a^2}{x}\right)=x^2(2a+R).$$

To find $\frac{dy}{dx}$ when $x=\infty$; it is easy to see that if $m-n$ be >1, supposing $y=x^{m-n}\frac{A+R}{B+R'}$, then $\frac{dy}{dx}=\infty$ when $x=\infty$; if $m-n=1$, then $\frac{dy}{dx}=\frac{A}{B}$ when $x=\infty$; and if $m-n$ be <1, then $\frac{dy}{dx}=0$ when $x=\infty$. So then we may very readily find $\frac{dy}{dx}$ at the same time that we find y corresponding to the value $x=\infty$.

For example, if $y=\frac{x^2}{a}\frac{x+a}{x-a}$ we may immediately put it in the form

x is $>-a$ and <0 it is $-$, when x is >0 and $<a$ it is $-$, when x is $>a$ and $<\infty$ it is $+$. Hence the table will be as follows.

x	y	$\frac{dy}{dx}$
	$+$	
$-a$	0	$-\frac{1}{2}$
	$-$	
0	0	0
	$-$	
a	∞	∞
	$+$	
∞	∞	∞

$$y = \frac{x^3\left(1+\frac{a}{x}\right)}{x\left(a-\frac{a^2}{x}\right)} = x^2\,\frac{1+R}{a+R'};$$

$$\therefore\ y=\infty,\ \text{ and } \frac{dy}{dx}=\infty,\ \text{ when } x=\infty.$$

For a second example, let $y = x\left(\frac{x+a}{x-a}\right)^2$;

$$\text{then } y = x\left(\frac{1+\frac{a}{x}}{1-\frac{a}{x}}\right)^2 = x\,\frac{1+R}{1+R'};$$

$$\therefore\ y=\infty,\ \text{ and } \frac{dy}{dx}=1,\ \text{ when } x=\infty;$$

For a third example, let $y = \dfrac{a^2\sqrt{x^2-a^2}}{(x-b)(x-c)}$;

$$\text{then } y = \frac{xa^2\sqrt{1-\frac{a^2}{x^2}}}{x^2\left(1-\frac{b}{x}\right)\left(1-\frac{c}{x}\right)} = \frac{1}{x}\cdot\frac{a^2+R}{1+R'};$$

$$\therefore\ y=0,\ \text{ and } \frac{dy}{dx}=0,\ \text{ when } x=\infty.$$

From this table it is immediately evident that fig. 22 represents the curve. For if we take AX, AY as axes, $AB = a$, $AC = -a$; then the table shews that at the point C the curve passes from the positive to the negative side of the axis of x at an angle $\tan^{-1}(-\frac{1}{2})$, as is represented in the figure; that it touches the axis of x at A, but does not cross it; that it goes off to an infinite distance when $x = AB$; and when x becomes greater than AB it appears on the positive side of the axis of x, evidently in the form FPG, for y is infinite when $x = AB$, becomes and continues positive when x is greater than AB, and becomes infinite again when x is infinite; y therefore must decrease as x gets greater than AB as far as a certain point after which it must increase again in order that it may go off to infinity when x becomes infinite without becoming negative; from which, it is evident that the curve must be in the form FPG. Since y and $\frac{dy}{dx}$ are both infinite when $x = \infty$, they must be very large when x is either a very large positive or a very large negative quantity, and therefore the curve must run off to infinity on the positive and negative side of the axis of y in the manner represented in the figure.

Representation by figures of different cases which may occur.

219. In the plates marked M the Student will find some of the different cases which may occur in tracing a curve represented by figures, to which he will find it useful to refer. Under each figure is set down the corresponding line in the table from which the figure is deduced. By means of this plate the use of the table will appear evident.

To determine the form of the curve more precisely, we must examine whether it has any points of contrary flexure.

220. Thus we are able to trace the *general* form of the curve; but this method does not always shew us the points of contrary flexure of the curve; to determine which we must resort to the method given in (180). Thus in the present case, differentiating y twice, we get

$$\frac{d^2y}{dx^2} = 2\frac{(x-a)^3 + 2a^3}{(x-a)^4},$$

the only real values of x which make this $= 0$ or ∞ are $x = a$, $x = a - 2^{\frac{1}{3}}a$ or $-a(2^{\frac{1}{3}} - 1)$: so that there can be only two

points of contrary flexure; which indeed the figure shews, for there must evidently be a change of flexure somewhere between C and A, since the concavity of the curve is turned upward at C and downward at A; also there is evidently a change of flexure at B. There is no change of flexure elsewhere, and therefore the figure represents the precise form of the curve.

221. In order to determine still more minutely the form of the curve, we may find the points where the ordinate is a maximum or a minimum; to do this in the present case, we have

We may find the maximum and minimum ordinates, in order to determine the form of the curve more minutely.

$$\frac{dy}{dx} = 2\frac{x^3 - ax^2 - a^2x}{a(x-a)^2}.$$

and therefore, when $\dfrac{dy}{dx} = 0$, we have

$$x^3 - ax^2 - a^2x = 0,$$

which gives $x = 0$, and $x = \dfrac{a}{2} \pm \sqrt{\dfrac{a^2}{4} + a^2}$, or $\dfrac{a}{2}(1 \pm \sqrt{5})$;

which three values correspond to the points A, Q, and P.

222. When the curve has asymptotes, it is often useful to find them; for they enable us to see more clearly the nature of the infinite branches of the curve; for example, take the equation

Asymptotes are useful in giving us an idea of the manner in which the branches of a curve go off to infinity. Example 2.

$$y = x\frac{(x+a)^2}{(x-a)^2}.$$

Forming a table as in the former case, we find

x	y	$\frac{dy}{dx}$
	$-$	
$-a$	0	0
	$-$	
0	0	1
	$+$	
a	∞	∞
	$+$	
∞	∞	1

$$\text{Since}^* \; y = x\left(\frac{1+\frac{a}{x}}{1-\frac{a}{x}}\right)^2 = x\left(1+\frac{4a}{x}+\&c.\right) = x + 4a + R,$$

where R becomes 0 when $x = \infty$, the curve has an asymptote whose equation is

$$y = x + 4a.$$

Hence, take AC (fig. 23) $= a$, $AB = a$, $AN = 4a$, and draw the line $DMNG$, making $\angle 45^\circ$ with AX, and the line BF perpendicular to AX. Then $DMNG$ is an asymptote to the curve; the curve comes from infinity below the axis of x, touches it at C, but does not cross it, crosses it at A at an angle 45°, goes off to infinity as it approaches the line BF, appears on the positive side of AX when it passes BF, and then turns off towards the line $DMNG$ as its asymptote.

(For more Examples, see Appendix S.)

How to trace a curve from its polar equation

223. We may trace the general form of a curve referred to polar co-ordinates in a similar manner, by putting down the values of θ which make r zero or infinity; but as there are often no such values, or only very few of them, a table of those values is not in general sufficient to indicate the form of the curve. We may in most cases perceive the form of the curve by considering whether r increases or diminishes as it turns round; and this we may see, either by simple inspection, or by finding $\frac{dr}{d\theta}$ and examining whether its sign is + or −, which will tell us whether r increases or diminishes with θ.

There is no use however in putting down the values of $\frac{dr}{d\theta}$ as we did those of $\frac{dy}{dx}$ in the case of rectangular curves: but it will sometimes be advisable to put down the signs which

* It very often happens, as in the present example, that we can immediately put y in the form $Ax + B + R$, where R becomes 0 when $x = \infty$; in such a case we have no need to resort to the general method of finding asymptotes given in the preceding Chapter.

$\frac{dr}{d\theta}$ has, in order to see whether r is increasing or diminishing with θ. When r is negative, $\frac{dr}{d\theta} = +$, indicates a diminution in the absolute numerical value of r, and *vice versâ*.

It is sometimes necessary, in order to make out properly the nature of the curve, to put down a few of the values of r corresponding to such values of θ, as $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ ... $-\frac{\pi}{2}$, $-\frac{3\pi}{2}$, ... &c.

When $\theta = \alpha$ makes $r = 0$, it is clear that there is a tangent at the origin making an angle α with the prime radius. When $\theta = \alpha$ makes $r = \infty$, we must find the corresponding asymptote.

It is usual to neglect the negative values of r in tracing polar curves, but there is no reason whatever to justify this omission, and we shall therefore always consider these values to be, as they really are, just as important as the positive values.

224. In measuring positive and negative values along r, it will save mistakes, to suppose an arrow-head fixed upon the radius vector to revolve with it as θ increases; then this arrow-head, supposing it to point in the positive direction when $\theta=0$, will always continue to point in the positive direction, whatever be the value of θ. Thus, in fig. 24, let $\theta = \frac{\pi}{4}$, and then from S to P is the positive direction, and from S to P' the negative: in fig. 25, let $\theta = \pi + \frac{\pi}{4}$, and then from S to P is the positive direction: in fig. 26, let $\theta = \frac{3\pi}{2} + \frac{\pi}{4}$, and then from S to P is the positive direction: and in fig. 27, let $\theta = -\frac{\pi}{4}$, and then from S to P is the positive direction.

Positive and negative values of r, how measured.

Thus we see that the positive direction with reference to a revolving line is not a fixed direction in space, but depends on

the angle at which the radius vector is inclined to its original position. The principle on which what we have just stated depends belongs to a different part of elementary mathematics; and therefore we assume it here without explanation.

Representation by figures of different cases.

Bearing these considerations in mind, it will in general be easy to trace the form of a curve from its polar equation, as will appear by the following examples. The Student will find in the plates marked M a representation by figures of some of the different cases which may occur, and the corresponding line of the table under each: to which he will find it useful to refer.

Example 1.

225. Let $r = a\theta$ be the equation to the curve.

Here then is no use in forming a table; we see immediately that as θ increases, r continually increases; when $\theta = 0$, $r = 0$; and when θ is negative, r is negative.

Hence the curve passes through the pole and touches the prime radius when $\theta = 0$, and then continually recedes from the pole as r turns round. When $\theta =$ a negative angle, ASP suppose (fig. 28), then r is negative, and therefore must be measured in the direction SP'. Hence the curve is evidently of the form represented in the figure, the dotted part corresponding to the negative values of θ being similar to that corresponding the positive.

Example 2.

226 Let $r = a\,\dfrac{\theta^2 - \pi^2}{\theta}$.

Here $\dfrac{dr}{d\theta} = a\left(1 + \dfrac{\pi^2}{\theta^2}\right)$ which is essentially positive: hence r always increases with θ.

When $\theta = 0$, $r = \infty$, and by the method explained in the last chapter $y = a\pi^2$ is the equation to the corresponding asymptote.

Forming a table, we have

θ	r	$\frac{dr}{d\theta}$
	$-$	$+$
$-\pi$	0	
	$+$	$+$
0	∞	...... asymp. $y = -a\pi^2$
	$-$	$+$
π	0	
	$+$	$+$

θ	r
$\frac{\pi}{2}$	$-\frac{3\pi}{2}a$
$\frac{3\pi}{2}$	$\frac{5\pi}{6}a$
....	....

Hence the curve has the form represented in fig. 29; the dotted part corresponding to the negative values of θ, and

$$SB = a\pi^2, \quad SC = \frac{3\pi a}{2}, \quad SD = \frac{5\pi a}{6}.$$

A reference to the plates M will help the Student to make out the forms of curves from tables of the corresponding values of the co-ordinates.

The curves whose equations are

$$y^2(x^2 - 4a^2) = x^2(x^2 - a^2) \ldots\ldots (42),$$

$$y^2(x^2 - 4a^2) = (x^2 - a^2) \ldots\ldots\ldots (43),$$

$$r = ae^\theta \sin\theta \ldots\ldots\ldots\ldots\ldots\ldots (44),$$

$$r = ae^\theta \sin 2\theta \ldots\ldots\ldots\ldots\ldots\ldots (45),$$

$$r = ae^\theta \sin\frac{\theta}{2} \ldots\ldots\ldots\ldots\ldots\ldots (46),$$

$$r = a\sin 3\theta \ldots\ldots\ldots\ldots\ldots\ldots (47),$$

$$r = a\sin\frac{\theta}{2} \ldots\ldots\ldots\ldots\ldots\ldots (48),$$

$$r = a\sin\frac{\theta}{3} \ldots\ldots\ldots\ldots\ldots\ldots (49),$$

are respectively represented by figures 42, 43, 44, &c...&c.

CHAPTER XVI.

SINGULAR POINTS OF CURVES.

In the preceding Chapter we shewed how the *general* form of a curve may be traced: we now proceed to shew how its remarkable points may be detected and examined.

Proposition. 227. *To investigate the nature of a curve in the immediate vicinity of any proposed point of it.*

Let the equation to the curve be

$$y = f(x);$$

let the abscissa of the proposed point be $x = a$, and let x_1 be any value of x a little less than a, and x_2 a little greater. Then we shall consider several different cases which may occur.

Case 1. 228. *We shall in the first place suppose that* $f(x_1)$ *and* $f(x_2)$ *have one real value each, and only one, and this being assumed:*

(1) Let $f(a) = \infty$, $f(x_1) =$ positive, $f(x_2) =$ positive,

then the ordinate belonging to the abscissa a is an asymptote, and the curve lies above the axis of x on each side of it: therefore fig. 50 represents the curve, where AM is the abscissa a, and $MP (= \infty)$ the corresponding ordinate.

If $f(x_1) =$ negative, and $f(x_2) =$ negative, fig. 51 represents the curve.

If $f(x_1) =$ negative and $f(x_2) =$ positive, then the curve lies below the axis of x on the left side of MP, and above it on the right, and therefore fig. 52 represents it.

If $f(x_1)$ = positive, and $f(x_2)$ = negative, fig. 53 represents the curve.

(2) Let $f(a)$ be a finite quantity; take $MP = f(a)$, fig. 54, and draw the line $O'PO$ parallel to AM; then $f(x) - f(a)$ represents the distance of any point on the curve above $O'PO$. Hence

If $f'(a) = \infty$, $f(x_1) - f(a)$ = positive, $f(x_2) - f(a)$ = positive,

the tangent at P coincides with MP, but the curve lies above OPO on each side of MP; therefore fig. 54 represents its form.

If $f'(a) = \infty$, $f(x_1) - f(a)$ = negative, and $f(x_2) - f(a)$ = negative; then fig. 55 represents the curve.

If $f'(a) = \infty$, $f(x_1) - f(a)$ = negative, and $f(x_2) - f(a)$ = positive; then the curve touches MP at P, lies below it on the left side of MP, and above it on the right; therefore fig. 56 represents it.

If $f'(a) = \infty$, $f(x_1) - f(a)$ = positive, and $f(x_2) - f(a)$ = negative then fig. 57 represents the curve.

If, however, $f'(a)$ be a finite quantity, draw the line $T'PT$ (fig. 58) making an angle $\tan^{-1} f'(a)$ with the axis of x; then this line is the tangent to the curve at P, and, its equation being $y' - f(a) = f'(a)(x - a)$,

$$y - y' \text{ or } f(x) - f(a) - f'(a)(x - a)$$

is the distance of any point of the curve above this line. Hence, if for brevity, we put

$$\phi(x) = f(x) - f(a) - f'(a)(x - a),$$

it is evident that:

If $\phi(x_1)$ = positive, and $\phi(x_2)$ = positive, the curve lies above the tangent $T'PT$ on both sides of MP, and is therefore represented by fig. 59.

If $\phi(x_1)$ = negative, and $\phi(x_2)$ = positive, the curve lies below $T'PT$ on the left side of MP, and above it on the right, and is therefore represented by fig. 60.

If $\phi(x_1)$ = positive, and $\phi(x_2)$ = negative, fig. 61 represents the curve.

These are all the cases that can occur when y has one real value, and only one, for each value of x.

Case 2. 229. Let us now in the second place suppose that $f(x_1)$ has no real value, and $f(x_2)$ two real values, and only two; and this being assumed:

(1) Let $f(a) = \infty$, and both the values of $f(x_2)$ positive: then MP is an asymptote to two branches of the curve, both on the right side of MP; therefore fig. 62 represents the curve.

If both values of $f(x_2)$ be negative, fig. 63 represents the curve.

If one value be positive, and the other negative, fig. 64.

(2) Let $f(a)$ be a finite quantity, and $f'(a) = \infty$, then:

If both values of $f(x_2) - f(a)$ be positive, two branches of the curve touch MP at P, but do not go below $O'PO$, nor appear on the left side of MP: therefore fig. 65 represents the curve.

If both values be negative, fig. 66.

If one value be positive and the other negative, fig. 67.

(3) Let $f(a)$ be a finite quantity, and $f'(a)$ be so also, then:

If both values of $\phi(x_2)$ be positive, two branches of the curve touch the line $T'PT$ at P, but do not go below it, nor appear on the left side of MP: therefore fig. 68 represents the curve.

If both values be negative, fig. 68, bis.

If one value be positive, and the other negative, fig. 69.

If we suppose that $f(x_2)$ has no real value, and $f(x_1)$ two real values and only two, then it is evident the curve in each case will be exactly similar in form to what it is when $f(x_1)$

has no real value, and $f(x_2)$ two real values; it will be merely reversed in position: thus, instead of fig. 68, we shall have fig. 70; and similarly the other figures.

These are all the cases that can occur when $f(x)$ has two real values, and only two, for some values of x, and no real values for others.

230. It sometimes happens that $f(x_1)$ and $f(x_2)$ are both impossible, no matter how near x_1 and x_2 may be taken to a, and yet $f(a)$ a real quantity; in such a case the point P is a point belonging to the curve, since its co-ordinates satisfy the equation to the curve, and we define the curve to be the assemblage of all the points whose co-ordinates satisfy that equation: but no points in the immediate vicinity of P belong to the curve; P therefore is an isolated point of the curve, completely detached from the other points. Such a point is usually called a *conjugate point*. **Conjugate points.**

231. If $f(x_1)$ and $f(x_2)$ have each several values while $f(a)$ has only one value, then several branches of the curve must meet at the point P; in such a case P is called a *multiple point*. **Multiple points.**

We shall not extend this enumeration of cases any farther, as it will be easy, after what has been explained, to make out the form of the curve in any case that may present itself.

232. The following are examples of the cases that we have just discussed. **Examples.**

(1) $y = a + \dfrac{a^3}{(x-a)^2}$.

Here $f(a) = \infty$, $f(x_1) =$ positive, $f(x_2) =$ positive; therefore fig. 50 represents the curve.

(2) $y = \dfrac{x-2a}{(x-a)^2}$.

Here $f(a) = \infty$, $f(x_1) =$ negative, $f(x_2) =$ negative; (fig. 51).

(3) $y = \dfrac{a^2}{x - a}$.

Here $f(a) = \infty$, $f(x_1) =$ negative, $f(x_2) =$ positive; (fig. 52).

(4) $y = a + a^{\frac{1}{3}} (x - a)^{\frac{2}{3}}$.

Here $f(a) = a$, $f'(a) = \infty$, $f(x_1) - f(a) =$ positive; $f(x_2) - f(a) =$ positive; (fig. 54).

(5) $y = a + a^{\frac{2}{3}} (x - a)^{\frac{1}{3}}$; (fig. 55).

(6) $y = x + \dfrac{(x - a)^2}{a}$.

Here $f(a) = a$, $f'(a) = 1$, $\phi(x_1) =$ positive, $\phi(x_2) =$ positive; (fig. 59).

(7) $y = x + \dfrac{(x - a)^3}{a^2}$; (fig. 60).

(8) $\{y (x - a) - a^2\}^2 = a (x - a)^3$.

Solving this, we have $y = \dfrac{a^2}{x - a} \pm \sqrt{ax - a^2}$.

Here therefore $f(a) = \infty$, $f(x_1)$ is impossible, and $f(x_2)$ has two real values both positive*; (fig. 62).

(9) $(y - a)^2 (ax - a^2) = a^4$.

Solving this, $y = a \pm \dfrac{a^2}{\sqrt{ax - a^2}}$.

Here $f(a) = \infty$, $f(x_1)$ is impossible, and $f(x_2)$ has two real values, one positive and the other negative; (fig. 64).

(10) $(y - a)^2 = ax - a^2$.

Solving this $y = a \pm \sqrt{ax - a^2}$,

* For when x nearly $= a$ the first term is very great compared with the second, and therefore y has the same sign as the first term.

$$\text{and } \frac{dy}{dx} = \pm \frac{a}{2\sqrt{ax - a^2}}.$$

Here $f(a) = a$, $f'(a) = \infty$, $f(x_1) - f(a)$ is impossible, and $f(x_2) - f(a)$ has two real values, one positive and the other negative; (fig. 67).

(11) $a(y - x)^2 = (x - a)^3$.

$$\text{Solving this } \quad y = x \pm \frac{(x - a)^{\frac{3}{2}}}{a^{\frac{1}{2}}},$$

$$\text{and } \frac{dy}{dx} = 1 \pm \tfrac{3}{2}\left(\frac{x - a}{a}\right)^{\frac{1}{2}}.$$

Here $f(a) = a$, $f'(a) = 1$, $\phi(x_1)$ is impossible, and $\phi(x_2)$ has two real values, one positive and the other negative; (fig. 69).

(12) $y = x + a^{\frac{2}{3}}(x - a)^{\frac{1}{3}} \pm a^{\frac{1}{2}}(x - a)^{\frac{1}{2}}$.

Here $f(a) = a$, $f'(a) = \infty$, $f(x_1)$ is impossible;

$$f(x) - f(a) = (x - a)^{\frac{1}{3}}\left\{a^{\frac{2}{3}} \pm a^{\frac{1}{2}}(x - a)^{\frac{1}{6}}\right\}.$$

Now $(x_2 - a)^{\frac{1}{6}}$ is very small, and $\therefore a^{\frac{2}{3}} \pm a^{\frac{1}{2}}(x_2 - a)$ is positive whether we take the + or the −. Hence $f(x_2) - f(a)$ has two real values both positive; (fig. 65).

(13) $y = x + \dfrac{(x - a)^2}{a} \pm \dfrac{(x - a)^{\frac{5}{2}}}{a^{\frac{3}{2}}}$.

Here $f(a) = a$, $f'(a) = 1$, $f(x_1)$ is impossible;

$$\phi(x) = \frac{(x - a)^2}{a}\left\{1 \pm \frac{(x - a)^{\frac{1}{2}}}{a^{\frac{1}{2}}}\right\}.$$

Now $(x_2 - a)^{\frac{1}{2}}$ is very small, and $\therefore 1 \pm \dfrac{(x_2 - a)^{\frac{1}{2}}}{a^{\frac{1}{2}}}$ is positive whether we take the + or the −; hence $\phi(x_2)$ has two real values both positive; (fig. 68).

(14) $y = a \pm (x - a)\sqrt{x - 2a}$.

Here $f(a) = a$, and $f(x_1)$ $f(x_2)$ are both impossible, since $\sqrt{x - 2a}$ is impossible for all values of x less than $2a$;

hence the point whose co-ordinates are $x = a$, $y = a$ is a conjugate point of this curve.

(15) $\{a(y-a) - (x-a)^2\}^2 = a^2(x-a)^2$.

Solving this $y = a \pm (x-a) - \dfrac{(x-a)^2}{a}$.

Here $f(x)$ has two different real values for every value of x except $x = a$; hence the curve has two branches which meet at the point whose co-ordinates are $x = a$, $y = a$. Since $f'(a) = \pm 1$ the tangents of the two branches at the point P make angles 45^0 and -45^0 with the axis of x; the branches therefore cross each other at right angles at the point P; the concavities of both turned downwards since $f^2(a)$ is negative; (fig. 71).

(16) $y = x + (x-a)\sin^{-1}\dfrac{x}{a}$. Here an infinite number of branches cross at the point $(x = a,\ y = a)$.

Cusps.

233. The curve at the point P in (fig. 54, 65) is called a *cusp* from its pointed form: in (fig. 69) it is called a cusp of the first kind, and in (fig. 68) a cusp of the second kind.

Singular points.

Points of contrary flexure, cusps, conjugate points, multiple points, &c. are called *singular points.*

How the existence of each of these singular points is indicated.

234. *The existence of a point of contrary flexure is indicated by* $\dfrac{d^2y}{dx^2}$ *changing its sign.* (*See* 180).

235. *The existence of a cusp is indicated by the curve not crossing a right line which does not coincide with the tangent.*

If $\dfrac{dy}{dx}$ be a finite quantity at a certain point, and if the curve does not cross the ordinate at that point; or, if $\dfrac{dy}{dx} = \infty$ at a certain point, and the curve does not cross the line drawn through that point parallel to the axis of x; then there is a cusp at that point.

236. To determine whether the cusp is of the first or second kind we must find $\phi(x)$ (see 228), and arrange it in ascending powers of $x - a$; suppose that the result of this is

How to determine whether the point is a cusp of the first kind, or not.

$$\phi(x) = A(x-a)^m + B(x-a)^n + \&c.$$
$$= (x-a)^m \{A + B(x-a)^{n-m} + \&c.\}$$

Now when $x - a$ is very small, $A + B(x-a)^{n-m} + \&c.$ will have the same sign as A, therefore $\phi(x_2)$ will have the same sign as $A(x_2 - a)^m$. Hence if m be a fraction with an even denominator (of course reduced to its lowest terms) $A(x_2 - a)^m$ will have two real values with opposite signs, and therefore the two values of $\phi(x_2)$ have opposite signs, and therefore the cusp is of the first kind. But if m be an integer or a fraction with an odd denominator $A(x_2 - a)^m$ has only one real value, and therefore both values of $\phi(x_2)$ have the same sign as A, and consequently the cusp is of the second kind represented by fig. 68, or 68 bis, according as A is positive or negative.

Of course we here suppose that $\phi(x_2)$ has two and only two real values, and $\phi(x_1)$ no real value. If it be $\phi(x_1)$ which has the two real values, all that we have just said is equally true; only the cusp in each case will be reversed in position.

We here suppose also that $f'(a)$ is not infinite. If it be, the cusp will be of the first kind, if $f(x_1) - f(a)$ and $f(x_2) - f(a)$ have each one real value of the same sign; and a cusp of the second kind, if $f(x_2) - f(a)$ has two real values, and $f(x_1) - f(a)$ no real value, or the reverse.

Cusps, when $f'(a) = \infty$, how distinguished.

237. If the quantities $f(a)$, $f'(a)$, $f^2(a)$, &c. have each one real and finite value; then when x is taken sufficiently near to a, we have

If $f(a)$, $f'(a)$, $f^2(a)$, &c. ......have each one real finite value, the point $\{x=a, y=f(a)\}$ cannot be a singular

$$f(x) = f(a) + f'(a)\frac{x-a}{\Gamma 1} + f^2(a)\frac{(x-a)^2}{\Gamma 2} + \dots \textit{ ad infinitum},$$

which gives one real value for each value of x.

point of any kind except one of contrary flexure.

Under such circumstances we have therefore the case where $f(a)$ and $f'(a)$ are both finite, and $f(x_1)$ and $f(x_2)$ have each one real value: the curve therefore must assume a form similar to one of those in fig. 72 in the immediate vicinity of the point P: consequently P may or may not be a point of contrary flexure, but it cannot be a cusp, or a multiple point, or a conjugate point.

Hence if $f(a)$, $f'(a)$, $f^2(a,)$ *&c. have each one real and finite value, the point whose co-ordinates are* a *and* $f(a)$ *must be either an ordinary point or a point of contrary flexure: it cannot be a singular point of any kind except one of contrary flexure.*

At all singular points except one of contrary flexure two conditions hold, when the equation to the curve is algebraical and cleared of radicals.

238. Suppose that the equation to the curve is an *algebraical* equation between x and y *cleared of radicals* of the form

$$A + Bx + Cy + Dx^2 + Exy + Fy^2 + Gx^3 \ldots + Py^p = 0,$$

which for brevity we shall represent by

$$U = 0 \ldots\ldots (1);$$

then to find the successive differential coefficients of y we have the following equations got by differentiating (1), viz.

$$d_y U \cdot \frac{dy}{dx} + d_x U = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$d_y U \cdot \frac{d^2y}{dx^2} + d_y^2 U \cdot \left(\frac{dy}{dx}\right)^2 + 2 d_x d_y U \cdot \frac{dy}{dx} + d_x^2 U = 0 \ldots (3),$$

$$d_y U \cdot \frac{d^3y}{dx^3} + d_y^3 U \cdot \left(\frac{dy}{dx}\right)^3 + \&\text{c.} \ldots = 0 \ldots\ldots (4),$$

&c. ...... &c. ......

....

From these equations we may find the values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ ... &c. corresponding to any values a and b of x and y which satisfy (1): for by substituting a and b for x and y

in (2), (3), (4), &c. we find the corresponding value of $\dfrac{dy}{dx}$ from (2), and then that of $\dfrac{d^2y}{dy^2}$ from (3), and then that of $\dfrac{d^3y}{dx^3}$ from (4), and so on.

Now it is evident from the form of the equations (1), (2), (3), (4), &c. that if d_yU is not zero, we thus obtain a single finite value for each of the quantities $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$, $\dfrac{d^3y}{dx^3}$ .... for then we evidently find

$$\frac{dy}{dx} = \frac{\text{a quantity which cannot be infinite*}}{d_yU}$$

$$\frac{d^2y}{dx^2} = \frac{\text{a quantity which cannot be infinite}}{d_yU},$$

and so on.

Hence it appears by the preceding Article that if d_yU be not zero, the point (ab) must be either an ordinary point or a point of contrary flexure. And the same may be proved in exactly the same manner of d_xU.

Hence at all singular points except a point of contrary flexure we must have

$$d_xU = 0, \quad d_yU = 0,$$

$U = 0$ *being an algebraical equation cleared of radicals.*

239. If therefore the equation to a curve can be reduced to the form of a rational and integral algebraical equation $U = 0$, and if we determine those points whose co-ordinates satisfy this equation, and moreover the equations $d_xU = 0$, $d_yU = 0$; then no other points of the curve but these can be singular points of any kind, except points of contrary flexure. **Hence we may find all the points of a curve which may be singular.**

To determine whether such points are really singular points, and the class to which they belong to, we must examine them separately; and first, we must determine how many

* The numerators of these fractions cannot be infinite, because U is a rational and integral function of x and y.

branches of the curve meet at each of these points, which we may do as follows.

To determine how many branches of a curve pass through a point at which $d_x U = 0$, $d_y U = 0$.

240. Let $x = a$ and $y = b$ be the co-ordinates of any one of these points; then since the substitution of a and b for x and y makes

$$d_x U = 0 \text{ and } d_y U = 0,$$

we must proceed to determine $\frac{dy}{dx}$ as in (153, &c.), and we shall arrive at an equation for determining $\frac{dy}{dx}$ of a higher order than the first. If this equation has n real roots, then n branches of the curve and no more meet at the point $(a b)$. If any of these roots be equal, then the corresponding branches have the same tangent at the point (ab). If when x is a little greater than a, y has n real values, and when x is a little less than a only $n - 2m$ real values, (for an *odd* number of real roots cannot disappear) then $2m$ of the branches do not appear on the left side of the ordinate b, and each two of these branches must therefore either touch the ordinate b as in fig. 67, which will be the case with those branches for which $\frac{dy}{dx}$ becomes ∞ when $x = a$; or else they must meet the point $(a b)$ in the form of a cusp as in (fig. 68) or (fig. 69), which will be the case with those branches for which $\frac{dy}{dx}$ does not become infinite when $x = a$.

If the $2m$ real values of y are wanting when x is a little greater than a, the same is true, only the cusps, &c. are reversed in position.

If the equation for determining $\frac{dy}{dx}$ have no real root $(a b)$ is a conjugate point.

The method of expansion in (145) will often en-

241. We may often very readily find the nature of a curve near a proposed point (ab) by expanding $y - b$ in powers of $x - a$, by the method given in (145): each different expansion

will indicate a different branch of the curve, and will shew how it lies near the proposed point. able us to make out the nature of a curve near a proposed point.

We may also by the same method make out the nature of the infinite branches of a curve, by putting $\frac{1}{x}$ for x, and expanding y in powers of x.

And thus we may often make out very readily the form of a curve when we cannot solve its equation. (For Examples see Appendix V.)

CHAPTER XVII.

THE GENERAL THEORY OF CONTACT. INTERSECTION OF CONSECUTIVE CURVES.

Different orders of contact.

242. LET PQ, PQ', PQ'', (fig. 80) be three curves having a common point P, the co-ordinates of which are $AM = a$, $MP = b$; and let the respective equations to these three curves be

$$NQ = y = f(x), \quad NQ' = y' = \phi(x), \quad NQ'' = y'' = \psi(x);$$

$$\text{then } \frac{QQ''}{QQ'} = \frac{\psi(x) - f(x)}{\phi(x) - f(x)}.$$

Which, since the three curves meet at P, and therefore

$$f(a) = \phi(a) = \psi(a),$$

assumes the form $\frac{0}{0}$ when $x = a$.

Hence, by (148), when x approaches a, we have in general,

$$\text{lim. val. of } \frac{QQ''}{QQ'} = \frac{\psi'(a) - f'(a)}{\phi'(a) - f'(a)}.$$

Now suppose that $\phi'(a) = f'(a)$, while $\psi'(a)$ does not $= f'(a)$; then this limiting value is infinite; therefore when the point N approaches M, QQ'' gets continually larger in comparison with QQ', and may be made as many times larger than it as we please by sufficiently diminishing MN; and therefore the curve PQ' must lie infinitely closer to PQ in the immediate vicinity of the point P than the curve PQ'' does.

Hence it appears that if, for the same abscissa, two curves have the same value, not only of y, but also of $\frac{dy}{dx}$, they not

only meet, but also lie infinitely closer to each other in the immediate vicinity of the point of occurse than they would do if they had not the same value of $\frac{dy}{dx}$ at that point.

But suppose that $\psi'(a) - f'(a)$, and $\phi'(a) - f'(a)$ are both zero, then by (148) we have in general, when x approaches a,

$$\text{lim. val. of } \frac{QQ''}{QQ'} = \frac{\psi^2(a) - f^2(a)}{\phi^2(a) - f^2(a)}.$$

Suppose here that $\phi^2(a) = f^2(a)$ while $\psi^2(a)$ does not $= f^2(a)$, then this limiting value is infinite. We may therefore conclude just as before, that if for the same abscissa two curves have the same values, not only of y and $\frac{dy}{dx}$, but also of $\frac{d^2y}{dx^2}$, then they lie infinitely closer to each other in the immediate vicinity of the point of occurse than they would do if they had not the same value of $\frac{d^2y}{dx^2}$.

And in general, if

$$\phi(a) - f(a), \quad \phi'(a) - f'(a) \ldots\ldots \phi^{n-1}(a) - f^{n-1}(a),$$
$$\psi(a) - f(a), \quad \psi'(a) - f'(a) \ldots\ldots \psi^{n-1}(a) - f^{n-1}(a),$$

be each zero, we have, when x approaches a,

$$\text{lim. val. of } \frac{QQ''}{QQ'} = \frac{\psi^n(a) - f^n(a)}{\phi^n(a) - f^n(a)}.$$

Suppose here that $\phi^n(a) = f^n(a)$ while $\psi^n(a)$ does not $= f^n(a)$; then this limiting value is infinite. We may therefore conclude *that if for the same abscissa two curves have the same values of* y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2} \cdots \frac{d^ny}{dx^n}$, *then they lie infinitely closer to each other in the immediate vicinity of the point of occurse than they would do if they had not the same value of* $\frac{d^ny}{dx^n}$.

Orders of contact depend upon the differential coefficients.

Hence it is that when two curves meet, and at the point of occurse have the same value of $\frac{dy}{dx}$, they are said to have contact of the first order, if moreover the same value $\frac{d^2y}{dx^2}$, contact of the second order, and so on; and in general, if they have the same values of $\frac{dy}{dx}, \frac{d^2y}{dx^2} \dots \frac{d^ny}{dx^n}$, they are said to have *contact of the* n[th] *order*.

Contact of an even order is accompanied with intersection, whereas contact of an odd order is not.

243. If $\phi(a) - f(a)$, $\phi'(a) - f'(a) \dots \phi^n(a) - f^n(a)$ be each zero, and $\phi^{n+1}(a) - f^{n+1}(a)$ not zero, then by (121), $\phi(x) - f(x)$ has the same sign as

$$\{\phi^{n+1}(a) - f^{n+1}(a)\}(x-a)^{n+1}$$

for all values of x sufficiently near a: now if the contact be of an even order n is even, and $n+1$ is odd, and therefore $\phi(x) - f(x)$ changes its sign when x passes through the value a; therefore, at one side of the point P, $\phi(x)$ is greater than $f(x)$, and at the other side less; i. e. the curves cross each other at P, as in fig. 31. But if the contact be of an odd order, n is odd, and $n+1$ even, and therefore QQ' does not change its sign when x passes through the value a; i. e. the curves meet each other without crossing at the point P, as in fig. 32. *Hence contact of an even order is accompanied with intersection, but contact of an odd order is not.*

What has been said is true for oblique and polar co-ordinates also.

244. It is evident that all we have just said is equally true whether the co-ordinates be rectangular or oblique. Moreover it is easy to see that if we refer to curves to any new axes of co-ordinates, the degree of contact is not altered.

The order of contact is not changed by a change of co-ordinates.

It is clear also that what we have said applies equally to polar co-ordinates; i. e. if $\frac{dr}{d\theta}, \frac{d^2r}{d\theta^2} \dots \frac{d^nr}{d\theta^n}$ have the same values in both curves at the point of occurse, the contact will be of the n^{th} order: for then it is easy to see, putting $x = r\cos\theta$, $y = r\sin\theta$, that $\frac{dy}{dx}, \frac{d^2y}{dx^2} \dots \frac{d^ny}{dx^n}$ will also have the same values in both curves.

245. To make a line whose equation is

Example 1. A right line.

$$y_1 = Ax + B \text{ .... } (1),$$

have contact of the first order with a given curve whose equation is $y = f(x)$, at a given point whose abscissa is a.

y must $= y_1$ when $x = a$; therefore we have

$$f(a) = Aa + B \text{ ...... } (2).$$

$\dfrac{dy}{dx}$ must $= \dfrac{dy_1}{dx}$ when $x = a$; therefore we have

$$f'(a) = A \text{ .... } (3).$$

(1), (2) and (3) give

$$y - f(a) = f'(a)(x - a),$$

which is the equation required. It coincides with the common equation to the tangent, see (173), and hence the ordinary tangent has contact of the first order with the curve.

If we wish to make the line (1) have contact of the second order with the curve, we must put $\dfrac{d^2y}{dx^2} = \dfrac{d^2y_1}{dx^2}$ when $x = a$: but this gives $f''(a) = 0$, and this equation of course cannot hold except at particular points of the curve. Hence a right line cannot have contact of an order higher than the first with a curve except at particular points of it, determined by the equation $\dfrac{d^2y}{dx^2} = 0$ (or in some cases $\dfrac{d^2y}{dx^2} = \infty$, see note *).

246. Suppose that the curve is referred to polar co-ordinates, and that its equation is $r = f(\theta)$. The equation to a right line being $y = \tan\alpha\,(x + a)$, putting $x = r_1 \cos\theta$, $y = r_1 \sin\theta$, becomes

Example 2. A right line referred to polar co-ordinates.

$$r_1 = \frac{a \sin\alpha}{\sin(\theta - \alpha)},$$

* When $\frac{d^2y}{dx^2} = \infty$ $\frac{d^2x}{dy^2}$ (y being the independant variable) will be 0 in certain cases, and then there will be contact of the second order.

$$\text{then } \frac{1}{r_1} = \frac{\sin(\theta - a)}{a \sin \alpha}, \quad \frac{1}{r_1^2}\frac{dr_1}{d\theta} = \frac{\cos(\theta - a)}{a \sin \alpha},$$

$$\text{and } \therefore \frac{dr_1}{d\theta} = -r_1 \cot(\theta - a).$$

Now suppose that the line has contact of the first order with the curve at the point whose polar abscissa is $\theta = \beta$, then when $\theta = \beta$ we must have $r_1 = r$ and $\dfrac{dr_1}{d\theta} = \dfrac{dr}{d\theta}$,

$$\therefore f(\beta) = \frac{a \sin \alpha}{\sin(\beta - a)}, \quad \text{and } f'(\beta) = -f(\beta)\cot(\beta - a).$$

From the second of these equations we may determine a, and then from the first a, and thus the polar equation to a line touching a curve given by a polar equation at any proposed point may be found.

Since β may have any value we have in general

$$f'(\theta) = -f(\theta)\cot(\theta - a),$$

$$\text{or} \quad \tan(a - \theta) = \frac{f(\theta)}{f'(\theta)} = \frac{r d\theta}{dr},$$

which coincides with the result obtained in (196). Since $a - \theta$ is evidently the angle under the tangent and radius vector, i.e. ϕ.

The order of contact which we may make two curves have with each other.

247. The order of contact which we may make two curves have with each other depends in general upon the number of constants we have at our disposal; if there be $n + 1$ constants disposable we may so determine them that the contact shall be of the n^{th} order; for then we may, by giving them proper values, satisfy the $n + 1$ equations

$$y_{,} = y, \quad \frac{dy_{,}}{dx} = \frac{dy}{dx}, \quad \frac{d^2y_{,}}{dx^2} = \frac{d^2y}{dx^2} \cdots \frac{d^n y_{,}}{dx^n} = \frac{dy^n}{dx^n},$$

which equations must be fulfilled to produce contact of the n^{th} order.

Thus in the most general equation to the right line, $y = Ax + B$, there are two disposable constants A and B, and

therefore we may in general make a right line have contact of the first order, but no higher, with a given curve at a given point of it. In the most general equation to a circle, viz. $(x - A)^2 + (y - B)^2 = R^2$ there are three disposable constants A, B and R, and therefore we may in general make a circle have contact of the second order, but no higher, with a given curve at a given point.

Example 3. A circle made to touch a proposed curve with contact of 2nd order.

248. To determine A, B, R, so that the circle

$$\{(x - A)^2 + (y_1 - \beta)^2 = R^2\}$$

shall have contact of the second order with the curve

$$\{y = f(x)\}$$

at the point whose abscissa is a.

We have

$$(x - A)^2 + (y_1 - B)^2 = R^2 \ldots (1),$$

$$x - A + (y_1 - B)\frac{dy_1}{dx} = 0 \quad \ldots (2),$$

$$1 + \left(\frac{dy_1}{dx}\right)^2 + (y_1 - B)\frac{d^2y_1}{dx^2} = 0 \quad \ldots (3).$$

Now since the circle touches the curve with contact of the second order at the point whose abscissa is a, if for brevity we put $f(a) = b$, $f'(a) = p$, $f''(a) = q$ must have

$$y_1 = b, \quad \frac{dy_1}{dx} = p, \quad \frac{d^2y_1}{dx^2} = q \quad \text{when } x = a,$$

and therefore putting $x = a$ in the equations (1), (2), (3), they become

$$(a - A)^2 + (b - B)^2 = R^2 \ldots (1),$$

$$a - A + (b - B)\,p = 0 \quad \ldots (2),$$

$$1 + p^2 + (b - B)\,q = 0 \quad \ldots (3).$$

$$(3) \text{ gives } b - B = -\frac{1 + p^2}{q},$$

then (2) gives $a - A = p\dfrac{1+p^2}{q}$,

and then (1) gives $R = \pm\dfrac{(1+p^2)^{\frac{3}{2}}}{q}$,

which equations give us B, A, and R, and thus the circle is completely determined.

Circle of curvature. Its radius is the index of curvature ρ. See (181.)

This circle is commonly called the osculating circle, or the circle of curvature. Since a may belong to any point whatever of the curve, we may put x instead of a, and $\therefore$ y, $\dfrac{dy}{dx}$, and $\dfrac{d^2y}{dx^2}$ instead of b, p, q respectively, and then we have

$$R = \pm\frac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}} = \pm\frac{ds^3}{dx\,d^2y};$$

hence it appears that $R = \rho$ (see 184), and hence the point P' (185) is the center of the osculating circle.

It follows therefore that the limiting position of the intersection of two consecutive normals to a curve when they approach each other is the center of the osculating circle.

We may also arrive at this result by means of the following article.

Of the ultimate intersection of curves.

249. Let the equations to two curves be

$$f(xya) = 0\ldots\ldots(1),$$

$$f(xya') = 0\ldots\ldots(2),$$

(2) differing from (1) only in having a' in place of a, a and a' being two constants to which we may assign any values we please. Then if we determine x and y from these two equations taken together, the resulting values will be the co-ordinates of a point of intersection of the two curves. Now this point, which we shall call P, must be some definite point as long as a' is different from a, but if a' becomes

equal to a, it ceases to be a definite point, since then the two curves are identical. x and y therefore become illusory when $a' = a$, but of course they have some limiting value when a' approaches a as in the example (22). We may determine the limiting values of x and y as follows.

By Lemma XXI. the equation (2) may be put in the form

$$f(xya) + \{f'(xy\underset{\cdot}{a}) + Q\}(a' - a) = 0, \quad \text{(see 77)}$$

where Q is a quantity which may be diminished *ad libitum* by sufficiently diminishing $a' \sim a$. In virtue of (1), and dividing out $a' - a$, this equation becomes

$$f'(xy\underset{\cdot}{a}) + Q = 0 \ldots\ldots (3).$$

Of course we necessarily suppose in this process that a' does not actually $= a$.

From (1) and (3) we may determine x and y, which will of course be different for different values of a'. Let a' approach a; then, in virtue of (1) and (3), the limiting values of $f(xya)$ and of $f'(xy\underset{\cdot}{a}) + Q$ must be zero; if therefore x_1 and y_1 be the limiting values of x and y, since that of Q is zero, we have by Lemma VIII.

$$f(x_1y_1a) = 0, \text{ and } f'(x_1y_1\underset{\cdot}{a}) = 0,$$

and from these equations we may determine x_1 and y_1.

Since x_1 and y_1 are the limiting values of x and y when a' approaches a, it is clear that we may diminish the distance between the points (xy) and (x_1y_1) *ad libitum* by sufficiently diminishing $a' - a$. The point x_1y_1 is therefore called the *ultimate intersection* of two consecutive positions of the curve (1), and its co-ordinates are the values of x and y which are obtained from the equation

$$f(xya) = 0,$$

and its partial derivative with respect to a, viz.

$$f'(xy\underset{\cdot}{a}) = 0;$$

a is called the variable parameter of the curve.

Example 1. **250.** **To find the ultimate intersection of two consecutive positions of a line which always includes equal areas between it and the co-ordinate axes.**

Let $\frac{x}{\alpha} + \frac{y}{\beta} = 1$ be the equation to the line, and $2c$ the constant area; then $\alpha\beta = c$, and the equation to the line becomes

$$\frac{x}{\alpha} + \frac{\alpha y}{c} - 1 = 0 \ldots\ldots (1),$$

α being the variable parameter.

The partial derivative of this with respect to α is

$$-\frac{x}{\alpha^2} + \frac{y}{c} = 0 \ldots\ldots (2),$$

from these equations we easily get

$$\frac{x}{\alpha} + \frac{x}{\alpha} - 1 = 0, \quad \text{or } x = \frac{\alpha}{2};$$

$$\text{and then } y = \frac{c}{2\alpha}.$$

The value $x = \frac{\alpha}{2}$ shews that the line is bisected by the point of ultimate intersection.

We may verify the correctness of this result thus.

Let α' be another value of α, then the equation to the corresponding line is

$$\frac{x}{\alpha'} + \frac{\alpha' y}{c} - 1 = 0 \ldots\ldots (3),$$

(3) α' − (1) α gives

$$(\alpha'^2 - \alpha^2)\frac{y}{c} - (\alpha' - \alpha) = 0, \quad \text{or } y = \frac{c}{\alpha' + \alpha};$$

and therefore the limiting value of y when α' approaches α is $\frac{c}{2\alpha}$, and therefore by (1) that of x is $\frac{\alpha}{2}$.

251. Hence we may find the ultimate intersection of two consecutive normals as follows. Example 2.

The equation to the normal to a curve $y = f(x)$ at the point whose abscissa is a is

$$f'(a)\{y - f(a)\} + x - a = 0 \ldots\ldots\ldots\ldots (1),$$

taking its partial derivative with respect to a we have

$$f''(a)\{y - f(a)\} - \{f'(a)\}^2 - 1 = 0 \ldots\ldots (2),$$

and the values of x and y got from these equations are the co-ordinates of the point required. These values are

$$y = f(a) + \frac{\{f'(a)\}^2 + 1}{f''(a)},$$

$$x = a - f'(a) \cdot \frac{\{f'(a)\}^2 + 1}{f''(a)}.$$

Comparing these values with those of A and B in (249) we see that the point of ultimate intersection of two consecutive normals is the center of the circle of curvature.

252. We may find the locus of all the points of ultimate intersection of the set of curves represented by the equation How to find the locus of the points of ultimate intersection of a set of curves.

$$f(xya) = 0,$$

by eliminating a between this equation and its partial derivative

$$f'(xy\dot{a}) = 0.$$

Let us take the first example, last article, in which Example.

$$f(xya) = \frac{x}{a} + \frac{ay}{c} - 1 = 0,$$

$$f'(xy\underset{\cdot}{a}) = -\frac{x}{a^2} + \frac{y}{c} = 0.$$

Eliminating a, we find

$$a = 2x, \quad \text{and} \; \therefore \; xy = c,$$

which gives a rectangular hyperbola referred to its asymptotes.

CHAPTER XVIII.

LAGRANGE'S THEOREM. ELIMINATION OF CONSTANTS AND FUNCTIONS BY DIFFERENTIATION. EXPANSION BY MEANS OF THIS ELIMINATION.

THE following theorem, due to Lagrange, is often found useful in expanding functions which are given by equations.

Lagrange's Theorem.

253. *Suppose that* u, y, z, x *are variables connected by the equations*

$$u = f(y) \;\ldots\ldots\; (1), \qquad y = z + x\phi(y) \;\ldots\ldots\; (2),$$

to expand u *in a series of powers of* x.

To do this, we shall find $d_x^n u$ as follows:

y is a function of the independant variables z and x, and we have, by differentiating (2) with respect to z and x,

$$d_x y = \phi(y) + x\phi'(y)\, d_x y; \quad \text{and } \therefore\ d_x y = \frac{\phi(y)}{1 + x\phi'(y)},$$

$$d_z y = 1 + x\phi'(y)\, d_z y; \qquad \text{and } \therefore\ d_z y = \frac{1}{1 + x\phi'(y)} \ldots (3);$$

$$\text{hence, } d_x y = \phi(y)\, d_z y \;\ldots\ldots\; (4);$$

$$\text{and } \therefore\ d_x u = f'(y)\, d_x y = f'(y)\,\phi(y)\, d_z y \;\ldots\ldots\; (5).$$

Now if $\psi(y)$ denote any function of y,

$$\begin{aligned} d_x\{\psi(y)\, d_z y\} &= \psi'(y)\, d_x y\, d_z y + \psi(y)\, d_x d_z y \\ &= d_z\{\psi(y)\, d_x y\} \\ &= d_z\{\psi(y)\,\phi(y)\, d_z y\}, \text{ by } (4). \end{aligned}$$

Hence supposing that $\psi(y) = f'(y)\,\phi(y)$, and differentiating (5), we have

$$d_x^2 u = d_x[f'(y)\{\phi(y)\}^2 d_z y],$$

and again $d_x^3 u = d_x d_x[\ \ldots\ldots\ldots\ldots\ldots\ ]$

$$= d_z^2[f'(y)\{\phi(y)\}^3 d_z y], \text{ similarly,}$$

and so on; and finally,

$$d_x^n u = d_z^{n-1}[f'(y)\{\phi(y)\}^n d_z y].$$

Now when $x = 0$, $z = y$, and by (3) $d_z y = 1$; therefore, when $x = 0$, we have

$$d_x^n u = d_z^{n-1}[f'(z)\{\phi(z)\}^n].$$

Hence, by Taylor's series, if for brevity we put $f'(z) = V$, $\phi(z) = Z$, we have $\{\text{since } u = f(z) \text{ when } x = 0\}$,

$$u = f(z) + VZ \cdot \frac{x}{\Gamma 1} + d_z(VZ^2) \cdot \frac{x^2}{\Gamma 2} + d_z^2(VZ^3)\frac{x^3}{\Gamma 3}\ldots$$

$$+ d_z^{n-1}(VZ^n)\frac{x^n}{\Gamma n} + \&c.\ldots$$

which is the developement required. The following example will shew the use of it.

Let $u = e^y$, $y = z + xe^y$. Example.

Here $f(y) = e^y$; $\therefore f'(z)$ or $V = e^z$,

$\phi(y) = e^y$; $\therefore \phi(z)$ or $Z = e^z$;

$$\therefore d_z^{n-1}(VZ^n) = d_z^{n-1}\{e^{(n+1)z}\},$$

$$= (n+1)^{n-1}e^{(n+1)z},$$

which $= 2^0 e^{2z}$, $3e^{3z}$, $4^2 e^{4z}$, $5^3 e^{5z}\ldots$ when $n = 1, 2, 3, 4, \ldots$ &c. respectively; hence

$$u = e^z + e^{2z}\frac{x}{\Gamma 1} + 3e^{3z}\frac{x^2}{\Gamma 2} + 4^2 e^{4z}\frac{x^3}{\Gamma 3} + 5^3 e^{5z}\frac{x^4}{\Gamma 4}\ldots + \&c.$$

254. When we are given an equation between x and y containing any constants a_1, a_2, $a_3 \ldots a_n$ suppose, we may eliminate these constants by differentiation, as follows. Let the given equation be (Elimination of constants by differentiation.)

$$U = 0 \ldots\ldots\ldots (1),$$

and let it be differentiated n times, and so give the following equations; viz.

$$dU = 0 \ldots\ldots\ldots (2),$$

$$d^2U = 0 \ldots\ldots\ldots (3),$$

$$\ldots\ldots\ldots$$

$$d^nU = 0 \ldots\ldots\ldots (n+1);$$

then from these $n+1$ equations we may in general eliminate the n quantities $a_1, a_2, a_3 \ldots a_n$, and so obtain an equation between x, y, $\frac{dy}{dx}, \frac{d^2y}{dx^2} \ldots \frac{d^ny}{dx^n}$, not containing any of these constants.

Example. Let the given equation be

$$y - ax^2 + abx = 0 \ldots\ldots\ldots (1),$$

differentiating twice, we have

$$p - 2ax + ab = 0 \ldots\ldots\ldots (2), \qquad \left\{ p = \frac{dy}{dx}, \right.$$

$$q - 2a = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3); \qquad \left\{ q = \frac{d^2y}{dx^2}. \right.$$

hence, $2a = q$, and therefore, by (2),

$$ab = qx - p;$$

and therefore, by (1),

$$y - \frac{qx^2}{2} + qx^2 - px = 0,$$

$$\text{or} \quad y + \frac{qx^2}{2} - px = 0;$$

which is an equation between x, y, p, and q, not containing the constants a and b.

A number of different equations may be formed in this way.

255. It is evident that there be $n+m$ constants a_1, $a_2 \ldots a_n$, $a_{n+1} \ldots a_{n+m}$ in the equation $U=0$, we may eliminate any n of them we please between the equations

$$u = 0,$$

$$du = 0,$$

$$d^2u = 0,$$

$$\cdots\cdots\cdots$$

$$d^nu = 0.$$

Now there are $\dfrac{\overline{m+n}\,.\,\overline{m+n-1}\ldots\overline{m+1}}{1\,.\,2\ldots\ldots\ldots\ldots\ldots m}$ (= N suppose) different combinations of the $m+n$ constants a_1, a_2 ... &c.; therefore there are N different ways in which we may form an equation containing x, y, $\dfrac{dy}{dx}\ldots\dfrac{d^ny}{dx^n}$ and m of the constants; and consequently we may in general obtain N different equations involving x, y, $\dfrac{dy}{dx}\ldots\dfrac{d^ny}{dx^n}$ and m of the constants from the given equation.

In the example just given, if we eliminate a between (1) and (2), we find Example.

$$a = \frac{p}{2x-b} \text{ from (2)};$$

and therefore, by (1),

$$y - \frac{px(x-b)}{2x-b} = 0 \ldots\ldots (5).$$

Again, if we eliminate b, we obtain

$$y - px + ax^2 = 0 \ldots\ldots\ldots (6),$$

and thus we deduce from (1) two different equations (5) and (6), each involving x, y, $\dfrac{dy}{dx}$, and one of the constants.

256. We may eliminate the constants a_1, a_2, $a_3 \ldots a_n$ from $u = 0$ somewhat differently, as follows. This elimination may be effected somewhat differently.

Solve the equation $u = 0$ for a_1, and let the result be $u_1 = a_1$, then, differentiating, we find

$$du_1 = 0, \text{ in which } a_1 \text{ does not appear.}$$

Solve this equation for a_2, and let the result be $u_2 = a_2$, and then differentiating, we find

$$du_2 = 0, \text{ in which neither } a_1 \text{ nor } a_2 \text{ appear};$$

and proceeding in this manner we may eliminate a_1, a_2, a_3 ... successively.

Example.

Thus in the example just considered, if we solve for a we find

$$\frac{y}{x^2 - bx} = a,$$

therefore differentiating we have

$$\frac{p\,(x^2 - bx) - y\,(2x - b)}{(x^4 - bx)^2} = 0,$$

$$\text{or } (x^2 - bx)\,p - y\,(2x - b) = 0,$$

which equation does not contain a.

Again, solving for b, we get

$$\frac{x^2 p - 2xy}{xp - y} = b\,;$$

and therefore differentiating we find

$$(x^2 q - 2y)(xp - y) - (x^2 p - 2xy)\,xq = 0,$$

$$\text{or } x^2 qy - 2y\,(xp - y) = 0,$$

$$\text{or } x^2 y = 2\,(xp - y) = 0,$$

which is the same equation as before.

This method is generally longer than the former.

Eliminations of functions from equations.

257. In the same way that we eliminate constants from equations between two variables x and y, we may eliminate functions from equations between three variables x, y, z.

For let $u = 0$ ... (1)

be an equation between x, y, and z, containing the n functions $\phi_1(v_1)$, $\phi_2(v_2)$, $\phi_3(v_3)$ ... $\phi_n(v_n)$, v_1 v_2 ... v_n being any functions of x y z. Then in virtue of (1) we may consider z as a function of the two independant variables x and y, and we have by successively differentiating with respect to x and y the following set of equations:

$$u = 0,$$

$$d_x u = 0, \quad d_y u = 0,$$
$$d_x^2 u = 0, \quad d_x d_y u = 0, \quad d_y^2 u = 0,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
&c. ...
$$d_x^m u = 0, \quad d_x^{m-1} d_y u = 0 \;\dots\dots\; d_y^m u = 0,$$

in all $1 + 2 + 3 \dots + m$, or $\dfrac{(m+1)m}{2}$ equations. Now in these equations we evidently have involved the functions

$$\begin{array}{llll} \phi_1(v_1), & \phi_2(v_2), & \dots\dots\dots\dots & \phi_n(v_n), \\ \phi'_1(v_1), & \phi'_2(v_2), & \dots\dots\dots\dots & \phi'_n(v_n), \\ \phi_1^m(v_1), & \phi_2^m(v_2), & \dots\dots\dots\dots & \phi_n^m(v_n), \end{array}$$

in all mn different functions.

Hence if $\dfrac{(m+1)m}{2}$ be $> mn$, or $= mn + 1$ at least, we shall be able to eliminate all these functions from the equations, and so form an equation between $x\ y\ z$, $d_x z$, $d_y z$, $d'_x z$, ... &c. ... $d_x^m z$, $d_x^{m-1} d_y z$ ... $d_y^m z$, containing no trace of the functional letters ϕ_1, ϕ_2, $\phi_3 \dots \phi_n$

If $n = 1$ the condition $\dfrac{(m+1)m}{2} > mn$ becomes

$$\frac{m+1}{2} > 1, \text{ and } \therefore\ m > 1.$$

If $n = 2$ it becomes

$$\frac{m+1}{2} > 2, \text{ and } \therefore\ m > 3,$$

and so on.

Hence to eliminate one function it will be necessary in general to go beyond the first order of differentiation; to eliminate two functions we must go beyond the third order; and so on. Often however it happens that, in consequence of the peculiar form of the equations, the elimination may be effected without proceeding so far; as the following examples will shew.

Example. Let the given equation be

$$u = z\phi(x+y) + xy = 0 \;\ldots\; (1),$$

then $$d_x u = p\phi(x+y) + z\phi'(x+y) + y = 0 \;\ldots\; (2),$$

$$d_y u = q\phi(x+y) + z\phi'(x+y) + x = 0 \;\ldots\; (3),$$

where $p = d_x z$,

$q = d_y z$,

(2) and (3) give

$$(p-q)\,\phi(x+y) + y - x = 0,$$

which by (1) give

$$z\frac{x-y}{p-q} + xy = 0.$$

Thus we obtain from (1), by differentiation as far as the first order, an equation which contains no trace of ϕ.

Example 2. Let the given equation be

$$\phi(x+z) + \psi(y+z) = 0 \;\ldots\; (1),$$

then $$(1+p)\,\phi'(x+z) + p\,\psi'(y+z) = 0 \;\ldots\; (2),$$

$$q\,\phi'(x+z) + (1+q)\,\psi'(y+z) = 0 \;\ldots\; (3),$$

(2) and (3) give immediately

$$\frac{1+p}{q} = \frac{p}{1+p} \quad\text{or}\quad 1+p+q = 0,$$

from which ϕ and ψ are eliminated.

Developement may be effected by eliminating functions. Example $\cos(m\cos^{-1}x)$

258. This sort of elimination is often useful in expanding functions, as the following example will shew.

To develope $\cos(m\cos^{-1}x)$ in powers of x, assume

$$y = \cos(m\cos^{-1}x) \;\ldots\; (1),$$

then $$\frac{dy}{dx} = \frac{m}{\sqrt{1-x^2}}\sin(m\cos^{-1}x) \;\ldots\; (2);$$

$$\therefore\ \sqrt{1-x^2}\,\frac{dy}{dx} - m\sin(m\cos^{-1}x) = 0\,;$$

$\therefore$ differentiating again,

$$\sqrt{1-x^2}\,\frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}}\,\frac{dy}{dx} + \frac{m^2}{\sqrt{1-x^2}}\cos(m\cos^{-1}x) = 0\,;$$

or multiplying by $\sqrt{1-x^2}$, and putting y for $\cos(m\cos^{-1}x)$,

$$\frac{d^2y}{dx^2} - x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0 \;\dots\; (3).$$

Having thus eliminated the functions cos and $\cos^{-1}$ by differentiation, we shall be easily able to develope y in the powers of x by assuming

$$y = A_0 + A_1x + A_2x_2 \dots A_nx^n + A_{n+1}x^{n+1} + A_{n+2}x^{n+2} + \dots \text{\&c.}$$

Substituting this for y in (3) we find for the coefficient of x^n

$$(n+2)(n+1)A_{n+2} - n(n-1)A_n - nA_n + m^2A_n,$$

which being put equal to zero gives

$$A_{n+2} = \frac{n^2-m^2}{(n+1)(n+2)}A_n,$$

from which we evidently get

$$A_2 = \frac{-m^2}{\Gamma 2}A_0, \qquad A_4 = \frac{(-m^2)(2^2-m^2)}{\Gamma 4}A_0 \dots \text{\&c.},$$

$$A_3 = \frac{1^2-m^2}{\Gamma 3}A_1, \qquad A_5 = \frac{(1^2-m^2)(3^2-m^2)}{\Gamma 5}A_1 \dots \text{\&c.}$$

To determine A_0 and A_1, put $x = 0$, and $\therefore y = A_0$ and $\frac{dy}{dx} = A_1$, in (1) and (2), and we find

$$A_0 = \cos(m\cos^{-1}0) = \cos m\,\frac{2r+1}{2}\pi,$$

$$A_1 = m\sin m\,\frac{2r+1}{2}\pi.$$

Hence if for brevity we assume

$$U = 1 - \frac{m^2}{\Gamma 2}x^2 + \frac{m^2(m^2-2^2)}{\Gamma 4}x^4 - \frac{m^2(m^2-2^2)(m^2-4^2)}{\Gamma 6}x^6 \dots \&c.,$$

$$V = mx - \frac{m(m^2-1^2)}{\Gamma 3}x^3 + \frac{m(m^2-1^2)(m^2-3^2)}{\Gamma 5}x^5 \dots \&c.,$$

we have

$$\cos(m\cos^{-1}x) = U\cos\frac{2r+1}{2}m\pi + V\sin\frac{2r+1}{2}m\pi,$$

which is the general expansion of the cosine of a multiple arc in powers of the cosine of the arc.

CHAPTER XIX.

MISCELLANEOUS THEOREMS.

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259. If $y = uv$, u and v being two functions of x, then Theorem of Leibnitz. If $y = uv$ to find $d^n(uv)$.

$$d^n y = d^n u \cdot v + \frac{n}{1} d^{n-1} u \cdot dv + \frac{n(n-1)}{\Gamma 2} d^{n-2} u \cdot d^2 v \dots + u d^n v.$$

For we have

$$dy = du \cdot v + u \cdot dv,$$

$$d^2 y = d^2 u \cdot v + 2 du \cdot dv + u d^2 v,$$

$$d^3 y = d^3 u \cdot v + 3 d^2 u \cdot dv + 3 du \cdot d^2 v + u d^3 v,$$

....

Let us assume that

$$d^n y = d^n v \cdot u + A_n d^{n-1} v \cdot du + B_n d^{n-2} v \cdot d^2 u \dots\dots$$

then we easily find by differentiating,

$$d^{n+1} y = d^{n+1} v \cdot u + (A_n + 1)\, d^n v\, du + (B_n + A_n)\, d^{n-1} v\, d^2 u \dots;$$

$$\therefore\ A_{n+1} = A_n + 1, \quad B_{n+1} = B_n + A_n, \quad C_{n+1} = C_n + B_n \dots \&c.;$$

which shews that if the coefficients of $d^n y$ be the same as those of $(1 + x)^n$ expanded, the same will be true when $n + 1$ is put for n; but we have shewn that this law holds when $n = 1$ or 2 or 3; therefore it is true in general, and we have

$$d^n y = d^n u \cdot v + \frac{n}{1} d^{n-1} u \cdot dv + \frac{n(n-1)}{\Gamma 2} d^{n-2} u \cdot d^2 v \dots + u d^n v.$$

This is called Leibnitz' Theorem, and is often of use in finding successive differential coefficients.

n^{th} differential coefficient of $\phi(a+bx+cx^2)$.

260. *To find the* n^{th} *differential coefficient of* $f(x)$ *if* $f(x) = \phi(a + bx + cx^2)$.

We have $f(x+h) = \phi(A + Bh + ch^2) \dots (1)$,

$$\text{if } A = a + bx + cx^2, \quad B = b + 2cx.$$

Now the general term of $\phi(A + Bh + ch^2)$ expanded in powers of $Bh + Ch^2$ is

$$\phi^r(A)\,\frac{(Bh + ch^2)^r}{\Gamma r},$$

and the general term of this expanded in powers of ch^2 is

$$\phi^r(A)\cdot\frac{B^{r-s}c^s}{\Gamma s \,.\, \Gamma(r-s)}\,h^{r+s}.$$

In this term let us suppose that $r + s = n$, and it becomes

$$\phi^r(A)\,\frac{B^{2r-n}c^{n-r}}{\Gamma(n-r)\,\Gamma(2r-n)}\,h^n = U_r h^n, \text{ suppose.}$$

Now s is evidently not $> r$, and not < 0, i.e. $n - r$ is not $> r$ and not < 0, i.e. r is not $< \frac{n}{2}$ and not $> n$.

Hence the sum of the terms formed by giving r all possible values is

$$(U_n + U_{n-1} + U_{n-2} \dots U_{\frac{n}{2}})\,h^n \text{ if } n \text{ be even};$$

$$\text{and } (U_n + U_{n-1} + U_{n-2} \dots U_{\frac{n+1}{2}})\,h^n \text{ if } n \text{ be odd};$$

and hence it follows that the coefficient of h^n in the second member of (1) expanded in powers of h is

$$U_n + U_{n-1} + U_{n-2} \dots \&\text{c. } U_{\frac{n}{2} \text{ or } \frac{n+1}{2}}.$$

Now the coefficient of h^n in the first member of (1) expanded is, by Taylor's Theorem, $\frac{1}{\Gamma n}f^n(x)$; hence we have

$$f^n(x) = \Gamma n\,(U_n + U_{n-1} + U_{n-2} \dots U_{\frac{n}{2} \text{ or } \frac{n+1}{2}}),$$

where

$$U_r = \phi^r\,(a + bx + cx^2)\,\frac{(b+2cx)^{2r-n}c^{n-r}}{\Gamma(n-r)\,\Gamma(2r-n)},$$

which gives us the differential coefficient required.

261. Let $\phi(a+bx+cx^2) = \sqrt{1+x+x^2} = (1+x+x^2)^{\frac{1}{2}}$, Example of this method.

then $\phi^r\,(a+bx+cx^2)$

$$= \tfrac{1}{2}\left(\tfrac{1}{2}-1\right)\left(\tfrac{1}{2}-2\right)(\dots)\left(\tfrac{1}{2}-r+1\right)(1+x+x^2)^{\frac{1}{2}-r}$$

$$= \frac{1.3.5\dots(2r-1)}{2^r}(-1)^{r-1}(1+x+x^2)^{\frac{1}{2}-r};$$

$$\therefore\ U_r = \frac{1.3.5\dots(2r-1)}{2^r}(-1)^{r-1}(1+x+x^2)^{\frac{1}{2}-r}\frac{(1+2x)^{2r-n}}{\Gamma(n-r)\,\Gamma(2r-n)};$$

$$\therefore\ \Gamma n\,U_n = \frac{1.3.5\dots(2n-1)}{2^n}(-1)^{n-1}(1+x+x^2)^{\frac{1}{2}-n}(1+2x)^n,$$

$$\Gamma n\,U_{n-1} = \frac{n\,(n-1)}{1}\cdot\frac{1.3.5\dots(2n-3)}{2^{n-1}}(-1)^{n-2}$$
$$\cdot(1+x+x^2)^{\frac{1}{2}-n+1}(1+2x)^{n-2},$$

$$\Gamma n\,U_{n-2} = \frac{n.(n-1)(n-2)(n-3)}{1.2}\cdot\frac{1.3.5\dots(2n-5)}{2^{n-2}}(-1)^{n-3}$$
$$(1+x+x^2)^{\frac{1}{2}-n+2}(1+2x)^{n-4}.$$

....

Hence, if for brevity we put $\dfrac{1+x+x^2}{(1+2x)^2} = z$, we have

$$f^n(x) = \frac{1.3.5\ldots(2n-1)}{2^n}(-1)^n \cdot \frac{(1+2x)^n}{(1+x+x^2)^{n-\frac{1}{2}}}$$

$$\left\{1 - \frac{n-1}{1}\,\frac{2n}{2n-1}\,z + \frac{(n-2)(n-3)}{1.2}\,\frac{2n(2n-2)}{(2n-1)(2n-3)}\,z^2 + \&c.\right\},$$

as far as $z^{\frac{n}{2}}$ or $z^{\frac{n-1}{2}}$.

Example 2. Let $\phi(a+bx+cx^2) = \log(a+bx+x^2)$;

then $\phi^r(a+bx+cx^2) = (-1)^{r-1}\,\Gamma(r-1)\,(a+bx+x^2)^{-r}$;

$$\therefore\ U_r = \frac{\Gamma(r-1).(-1)^{r-1}}{\Gamma(n-r).\Gamma(2r-n)}(a+bx+x^2)^{-r}(b+2x)^{2r-n};$$

$$\therefore\ f^n(x) = \Gamma n\,(-1)^{n-1}.\frac{(b+2x)^n}{(a+bx+x^2)^n}\left\{\frac{1}{n} - z + \frac{z^2}{2n-4}\ldots\right\}.$$

The Cycloid.

Generation of the cycloid. If a circle NPS (fig. 73) roll on a straight line AX, a point P in the circumference of it will trace out a curve of the form $APBX$, which is called a *Cycloid*. It is a curve of some importance in Mechanics.

Equation to cycloid. We may find its equation as follows:

Let A be the point on the line AX with which P originally coincided, take $AM\,(=x)$ $MP\,(=y)$ to be the co-ordinates of P; draw NC from the center C to the point of contact N of the circle, join CP and draw PO parallel to MN, let a denote the radius CP, and θ the angle NCP; then, since in the rolling of the circle every point of the arc PN has coincided with every point of the line AN, it is evident that

$$AN = \text{arc } PN = a\theta;$$

therefore we have

$$\left.\begin{aligned} x &= AN - MN = a\theta - a\sin\theta \\ y &= CN - CO = a - a\cos\theta \end{aligned}\right\}\ldots\ldots(1);$$

$$\therefore\ y = a\,\text{vers}\,\theta;\ \text{ and }\ \therefore\ \theta = \text{vers}^{-1}\frac{y}{a};$$

$$\text{also, } \sin\theta = \sqrt{1 - \left(\frac{a-y}{a}\right)^2} = \frac{1}{a}\sqrt{2ay - y^2};$$

$$\therefore\ x = a\,\text{vers}^{-1}\frac{y}{a} - \sqrt{2ay - y^2} \,\ldots\ldots\ldots (2),$$

which is the equation to the cycloid.

Equation transferred to vertex.

The point B at which $\theta = \pi$, which is called the vertex of the cycloid, is often taken for the origin, the line BY' parallel to AX for the axis of y, and the line BX' perpendicular to AX for the axis of x. We may easily find the equation to the cycloid referred to these axes by putting in the equations (1) $y' + AX'$ or $y' + \pi a$ for x, and $BX' - x'$ or $2a - x'$ for y, which gives us

$$y' + \pi a = a(\theta - \sin\theta),$$

$$2a - x' = a(1 - \cos\theta);$$

which, if we put $\pi + \theta'$ in the place of θ, become

$$\left.\begin{aligned} y' &= a(\theta' + \sin\theta') \\ x' &= a(1 - \cos\theta') \end{aligned}\right\} \ldots\ldots\ldots\ldots (3),$$

and therefore as before (suppressing the dashes),

$$y = a\,\text{vers}^{-1}\frac{x}{a} + \sqrt{2ax - x^2} \,\ldots\ldots\ldots\ldots (4),$$

which is the equation to the cycloid referred to the vertex.

Tangent or normal to a cycloid.

To determine the tangent and normal at P.

From the equations (1) we find

$$dx = a(1 - \cos\theta)\,d\theta, \quad dy = a\sin\theta\,d\theta;$$

$$\therefore\ -\frac{dx}{dy} = -\frac{a(1 - \cos\theta)}{a\sin\theta}$$

$$= -\frac{PM}{MN} = \tan PNX';$$

therefore the equation to the normal at P is

$$y_1 - y = \tan PNX' (x_1 - x),$$

which is the equation to the chord PN.

Hence the chord PN *is the normal at* P, *and therefore the chord* PS, *which is perpendicular to* PN, *is the tangent at* P.

Radius of curvature. To determine the index of curvature at P.

It follows from what has been just proved, that if ψ be the angle the tangent at P makes with the axis of x,

$$\psi = \frac{\pi}{2} - PSC = \frac{\pi}{2} - \frac{\theta}{2};$$

$$\therefore \rho = \frac{ds}{d\psi} = -\frac{2ds}{d\theta}.$$

$$\begin{aligned} \text{Now} \quad ds^2 &= dx^2 + dy^2 \\ &= a^2 \{(1 - \cos\theta)^2 + \sin^2\theta\} d\theta^2 \\ &= 2a^2 (1 - \cos\theta) d\theta^2 = 2ayd\theta^2; \end{aligned}$$

$$\therefore \rho^2 = 8ay,$$

which gives ρ.

Since $PN^2 = SN.NO = 2ay$, this gives

$$\rho = 2PN;$$

therefore, if we produce PN till $NT = PN$, T is the center of curvature corresponding to P.

Evolute. Let $AU (= \beta)$, $UT (= \alpha)$ be the co-ordinates of T, then

$$MN = NU = a \sin\theta,$$

$$\text{and } UT = PM = a (1 - \cos\theta);$$

$$\therefore \beta = a (\theta + \sin\theta),$$

$$\alpha = a (1 - \cos\theta),$$

$$\text{and } \therefore\ \beta = a \operatorname{vers}^{-1}\frac{\alpha}{a} + \sqrt{2a\alpha - \alpha^2}.$$

Comparing this with equation (4) we see that the locus of T, i.e. the evolute is a cycloid $ATVXW$, as represented in the figure, being equal in magnitude to the original cycloid, but having its position altered, its vertex being at A, and its cusp being at V.

From the equation (1) we have **Length of any arc of a cycloid.**

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{2a^2(1 - \cos\theta)\, d\theta^2}$$

$$= 2a \sin\frac{\theta}{2}\, d\theta = -4d\left(a\cos\frac{\theta}{2}\right);$$

$$\therefore\ d\left(s + 4a\cos\frac{\theta}{2}\right) = 0,$$

and $\therefore\ s + 4a\cos\frac{\theta}{2}$ must equal some constant, C suppose.

Now, assuming s to represent the arc AP it is evident that when $\theta = 0$, $s = 0$, therefore we have

$$0 + 4a = C, \quad \therefore\ C = a;$$

$$\therefore\ s = 4a\left(1 - \cos\frac{\theta}{2}\right),$$

which gives the length of the arc AP in terms of θ.

When P is at B, $\theta = \pi$, and $\therefore\ \cos\frac{\theta}{2} = 0$, hence the length of the arc AB is $4a$. **arc BP = 2 chord PS.**

$$\text{Arc } BP = \text{arc } AB - \text{arc } AP$$

$$= 4a - 4a\left(1 - \cos\frac{\theta}{2}\right),$$

$$= 4a\cos\frac{\theta}{2}.$$

Now chord $PS = 2a \cos PSN = 2a \cos \frac{\theta}{2}$;

$$\therefore \text{ arc } BP = 2 \text{ chord } PS.$$

$s'^2 = 8ax'$. If we denote arc BP by s', then since

$$\text{chord } PS = \sqrt{SN \, . \, OS} = \sqrt{2ax'},$$

we have

$$s'^2 = 8ax',$$

which is the relation between the length of the arc and the abscissa measured from the vertex.

APPENDIX.

(A.)

(1) $$f(x) = \frac{3x^2 - 5}{x^2 + 1} = \frac{\phi(x)}{\psi(x)} \text{ suppose,}$$

$$\text{then } \phi'(x) = 6x, \quad \psi'(x) = 2x;$$

$$\therefore f'(x) = \frac{6x(x^2 + 1) - 2x(3x^2 - 5)}{(x^2 + 1)^2},$$

$$= \frac{16x}{(x^2 + 1)^2}.$$

(2) $$f(x) = \frac{x^2 - x + 1}{x^2 + x + 1}, \quad f'(x) = \frac{2(x^2 + 1)}{(x^2 + x + 1)^2}.$$

(3) $$f(x) = \frac{x^2 + 1}{x^2 - 1} + \frac{x - 1}{x + 1} + x,$$

$$f'(x) = \frac{-4x}{(x^2 - 1)^2} + \frac{2}{(x + 1)^2} + 1 = \frac{x^4 - 8x + 3}{(x^2 - 1)^2}.$$

(4) $$f(x) = \frac{x^3}{x^2 + 1} + \frac{x^2}{x + 1} + x.$$

(5) $$f(x) = \frac{(x^2 - 1)(x + 2)}{x - 3} = \frac{\phi(x)}{x - 3} \text{ suppose,}$$

$$\text{then } f'(x) = \frac{\phi'(x)(x - 3) - \phi(x)}{(x - 3)^2},$$

$$\phi'(x) = 2x(x + 2) + (x^2 - 1) = 3x^2 + 4x - 1;$$

$$\therefore f'(x) = \frac{(3x^2 + 4x - 1)(x - 3) - (x^2 - 1)(x + 2)}{(x - 3)^2} = \frac{2x^3 - 7x^2 - 12x + 5}{(x - 3)^2}$$

(6) $$f(x) = \frac{(x-2)(x-3)}{(x-1)(x-4)}.$$

(7) $$f(x) = 2x^{\frac{3}{2}} - \frac{2}{x^{\frac{1}{2}}} + \frac{x^{\frac{1}{2}} - 1}{x^{\frac{1}{2}} + 1},$$

$$f'(x) = 3x^{\frac{1}{2}} + x^{-\frac{3}{2}} + \frac{x^{-\frac{1}{2}}}{x + 1 + 2x^{\frac{1}{2}}}.$$

(8) $$f(x) = \frac{x^n - a^n}{x^n + a^n}, \quad f'(x) = \frac{2nx^{n-1}a^n}{(x^n + a^n)^2}.$$

(B.)

(1) $$f(x) = \sqrt{z} = \phi(z), \quad z = 1 + e^x = \psi(x),$$

then $$f'(x) = \phi'(z)\,\psi'(x) = \frac{e^x}{2\sqrt{z}}.$$

(2) $$f(x) = \tan z = \phi(z), \quad z = m\tan^{-1}x = \psi(x),$$

then $$f'(x) = \phi'(z)\,\psi'(x) = \frac{1}{\cos^2 z}\cdot\frac{m}{1+x^2}.$$

(3) $$f(x) = \log z = \phi(z), \quad z = \sin x + \tan x = \psi(x),$$

then $$f'(x) = \phi'(z)\,\psi'(x) = \frac{1}{z}\left(\cos x + \frac{1}{\cos^2 x}\right).$$

(4) $$f(x) = \cos^{-1} z = \phi(z), \quad z = e^x \sin x = \psi(x),$$

then $$f'(x) = \phi'(z)\,\psi'(x) = -\frac{1}{\sqrt{1-z^2}}\cdot e^x(\sin x + \cos x).$$

(C.)

(1) $$y = \frac{x}{\sqrt{1-x^2}} = x(1-x^2)^{-\frac{1}{2}},$$

$$dy = dx\,.\,(1-x^2)^{-\frac{1}{2}} + x\,d\,(1-x^2)^{-\frac{1}{2}},$$

$$d(1-x^2)^{-\frac{1}{2}} = -\tfrac{1}{2}(1-x^2)^{-\frac{3}{2}}\,d(1-x^2) = x(1-x^2)^{-\frac{3}{2}}\,dx;$$

$$\therefore\ \frac{dy}{dx} = (1-x^2)^{-\frac{1}{2}} + x^2(1-x^2)^{-\frac{3}{2}} = \frac{1}{(1-x^2)^{\frac{3}{2}}};$$

Or thus:

$$\log y = \log x - \tfrac{1}{2}\log(1-x^2);$$

$$\therefore\ \frac{dy}{y} = \frac{dx}{x} - \frac{1}{2}\,\frac{-2x\,dx}{1-x^2} = \frac{1}{x}\cdot\frac{dx}{1-x^2};$$

$$\therefore\ \frac{dy}{dx} = \frac{y}{x}\,\frac{1}{1-x^2} = \frac{1}{(1-x^2)^{\frac{3}{2}}}.$$

(2) $$y = \log(x\sin x + \cos^2 x)$$

$$dy = \frac{d(x\sin x + \cos^2 x)}{x\sin x + \cos^2 x}$$

$$= \frac{dx\,.\sin x + x\cos x\,dx + 2\cos x\,d\cos x}{x\sin x + \cos^2 x};$$

$$\therefore\ \frac{dy}{dx} = \frac{\sin x(1-2\cos x) + x\cos x}{x\sin x + \cos^2 x}.$$

(3) $$y = (\tan x)^{\sec x};$$

$$\therefore\ \log y = \frac{1}{\cos x}\log\tan x,$$

$$\frac{dy}{y} = -\frac{d\cos x}{\cos^2 x}\log\tan x + \frac{1}{\cos x}\,\frac{d\tan x}{\tan x};$$

$$\therefore\ \frac{dy}{dx} = y\left\{\frac{a\sin x}{\cos^2 x}\log\tan x + \frac{1}{\sin x\cos^2 x}\right\}.$$

(4) $$y = \log\frac{x}{\sqrt{a^2+x^2}} = \log x - \tfrac{1}{2}\log(a^2+x^2);$$

$$\therefore\ dy = \frac{dx}{x} - \frac{1}{2}\,\frac{2x\,dx}{a^2+x^2};$$

$$\therefore\ \frac{dy}{dx} = \frac{a^2}{x(a^2+x^2)}.$$

(5) $$y = \tan^{-1}\left(\frac{\epsilon^x \sin x}{1 + \epsilon^x \cos x}\right);$$

then $\log \tan y = x + \log \sin x - \log(1 + \epsilon^x \cos x)$;

$$\therefore \frac{d \tan y}{\tan y} = \frac{dy}{\cos y \sin y} = dx + \frac{\cos x \, dx}{\sin x} - \frac{\epsilon^x (\cos x - \sin x)\, dx}{1 + \epsilon^x \cos x};$$

$$\therefore \frac{dy}{dx} = \cos y \sin y \, . \left\{\frac{\sin x + \cos x + \epsilon^x}{(1 + \epsilon^x \cos x) \sin x}\right\}.$$

Now $$\sin y = \frac{\tan y}{\sqrt{1 + \tan^2 y}} = \frac{\epsilon^x \sin x}{\sqrt{1 + 2\epsilon^x \cos x + \epsilon^{2x}}},$$

$$\cos y = \frac{1}{\sqrt{1 + \tan^2 y}} = \frac{(1 + \epsilon^x \cos x)}{\sqrt{1 + 2\epsilon^x \cos x + \epsilon^{2x}}};$$

$$\therefore \frac{dy}{dx} = \frac{\epsilon^x (\sin x + \cos x + \epsilon^x)}{1 + 2\epsilon^x \cos x + \epsilon^{4x}}.$$

Or thus:

$$dy = \frac{dz}{1 + z^2}, \text{ where } z = \frac{\epsilon^x \sin x}{1 + \epsilon^x \cos x},$$

and therefore,

$$dz = \frac{d(\epsilon^x \sin x) . (1 + \epsilon^x \cos x) - \epsilon^x \sin x \, d(\epsilon^x \cos x)}{(1 + \epsilon^x \cos x)^2}$$

$$= \frac{\epsilon^x (\sin x + \cos x)(1 + \epsilon^x \cos x) - \epsilon^x \sin x \, \epsilon^x (\cos x - \sin x)}{(1 + \epsilon^x \cos x)^2} dx;$$

$$\therefore \frac{dy}{dx} = \frac{1}{1 + z^2} \, \frac{\epsilon^x (\sin x + \cos x + \epsilon^x)}{(1 + \epsilon^x \cos x)^2}$$

$$= \frac{\epsilon^x (\sin x + \cos x + \epsilon^x)}{1 + 2\epsilon^x \cos x + \epsilon^{2x}}.$$

(6) $$y = \log\left(\frac{x - \sqrt{1 - x^2}}{x + \sqrt{1 - x^2}}\right)$$

$$= \log(x - \sqrt{1 - x^2}) - \log(x + \sqrt{1 - x^2});$$

$$\therefore\ dy = \frac{dx + \dfrac{x\,dx}{\sqrt{1-x^2}}}{x - \sqrt{1-x^2}} - \frac{dx - \dfrac{x\,dx}{\sqrt{1-x^2}}}{x + \sqrt{1-x^2}};$$

$$\therefore\ \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}\left(\frac{x+\sqrt{1-x^2}}{x-\sqrt{1-x^2}} + \frac{x-\sqrt{1-x^2}}{x+\sqrt{1-x^2}}\right)$$

$$= \frac{1}{\sqrt{1-x^2}}\left(\frac{1}{2x^2-1}\right).$$

(7) $$y = \cfrac{x^m}{1 - \cfrac{x^m}{1 - \cfrac{x^m}{1 \ldots}}}$$

then $y = \dfrac{x^m}{1-y}$; and $\therefore\ y - y^2 = x^m$;

$$\therefore\ dy - 2y\,dy = mx^{m-1}dx,$$

$$\frac{dy}{dx} = \frac{mx^{m-1}}{1-2y} = \cfrac{mx^{m-1}}{1 - \cfrac{2x^m}{1 - \cfrac{x^m}{1 - \cfrac{x^m}{\cdots}}}}$$

(8) $$y = \log(x + \sec x + \sin^{-1} x),$$

$$dy = \frac{1 + d\sec x + d\sin^{-1} x}{x + \sec x + \sin^{-1} x}.$$

(9) $$y = \tan^{-1}\frac{1}{\sqrt{1-x^2}};$$

$$\therefore\ \log\tan y = \log x - \tfrac{1}{2}\log(1-x^2),$$

$$\frac{dy}{\sin y\cos y} = \frac{dx}{x} + \frac{x\,dx}{1-x^2}.$$

(10) $$y = \log \sqrt{\frac{1 + \sin\theta}{1 - \sin\theta}}$$

$$= \tfrac{1}{2}\log(1 + \sin\theta) - \tfrac{1}{2}\log(1 - \sin\theta),$$

$$dy = \tfrac{1}{2}\frac{\cos\theta d\theta}{1 + \sin\theta} + \tfrac{1}{2}\frac{\cos\theta d\theta}{1 - \sin\theta},$$

$$\frac{dy}{dx} = \frac{\cos\theta}{2}\,\frac{2}{\cos^2\theta} = \frac{1}{\cos\theta}.$$

(11) $$y = \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right).$$

$$dy = \frac{d\left(\frac{\pi}{4} + \frac{x}{2}\right)}{\sin\left(\frac{\pi}{4} + \frac{x}{2}\right)\cos\left(\frac{\pi}{4} + \frac{x}{2}\right)},$$

and $\sin\left(\frac{\pi}{4} + \frac{x}{2}\right)\cos\left(\frac{\pi}{4} + \frac{x}{2}\right) = \tfrac{1}{2}\sin\left(\frac{\pi}{2} + x\right)$;

$$\therefore\ \frac{dy}{dx} = \frac{1}{\cos x}.$$

(12) $$y = \frac{1}{\sqrt{a^2 - b^2}}\cos^{-1}\left(\frac{b + a\cos x}{a + b\cos x}\right),$$

$$\log\cos ny = \log(b + a\cos x) - \log(a + b\cos x) \quad \{n = \sqrt{a^2 - b^2}\};$$

$$\therefore\ \frac{d\cos ny}{\cos ny} \text{ or } -\frac{u\sin ny dy}{\cos ny} = \frac{-a\sin x dx}{b + a\cos x} + \frac{b\sin x dx}{a + b\cos x}.$$

$$\therefore\ \frac{dy}{dx} = \frac{1}{n}\cot ny\sin x\left(\frac{a}{b + a\cos x} - \frac{b}{a + b\cos x}\right)$$

$$= \cot ny \sin x\,\frac{n}{(b + a\cos x)(a + b\cos x)}.$$

Now since $\cos ny = \dfrac{b + a\cos x}{a + b\cos x}$;

$$\therefore \cot ny = \frac{b + a\cos x}{\sqrt{(a + b\cos x)^2 - (b + a\cos x)^2}} = \frac{b + a\cos x}{n\sin x};$$

$$\therefore \frac{dy}{dx} = \frac{1}{a + b\cos x}.$$

(D.)

(1) $y = \epsilon^{mx}$, $\dfrac{d^n y}{dx^n} = m^n \epsilon^{mx}$.

(2) $y = x^m$, $\dfrac{d^n y}{dx^n} = m\,.\,(m-1)(\,\dots\,)(m-n+1)\,x^{m-n}$.

(3) $y = \log x$, $\dfrac{d^n y}{dx^n} = (-)^{n-1}\,.\,\Gamma(n-1)\,.\,x^{-n}$.

(4) $y = \sin(x + a)$, $\dfrac{d^n y}{dx^n} = \sin\left(x + a + \dfrac{n\pi}{2}\right)$.

(5) $y = \tan x$,

$$\frac{d^2 y}{dx^2} = \frac{2\sin x}{\cos^3 x},$$

$$\frac{d^3 y}{dx^3} = \frac{2}{\cos^2 x} + \frac{6\sin^2 x}{\cos^4 x}$$

$$= -\frac{4}{\cos^2 x} + \frac{6}{\cos^4 x}.$$

&c. &c.

Hence assume that in general

$$\frac{d^{2n-1}y}{dx^{2n-1}} = \frac{A_n}{\cos^2 x} + \frac{B_n}{\cos^4 x} + \frac{C_n}{\cos^6 x} \dots\dots$$

and we find

$$\frac{d^{2n}y}{dx^{2n}} = \frac{2A_n\sin x}{\cos^3 x} + \frac{4B_n\sin x}{\cos^5 x} + \frac{6C_n\sin x}{\cos^7 x} \dots$$

$$\frac{d^{2n+1}y}{dx^{n+1}} = \frac{2A_n}{\cos^2 x} + \frac{4B_n}{\cos^4 x} + \frac{6C_n}{\cos^6 x} \ldots$$

$$+ \frac{2.3A_n \sin^2 x}{\cos^4 x} + \frac{4.5B_n \sin^2 x}{\cos^6 x} + \frac{6.7C_n \sin^2 x}{\cos^8 x} \ldots$$

$$= \frac{-2^2 A_n}{\cos^2 x} + \frac{2.3A_n - 4^2 B_n}{\cos^4 x} + \frac{4.5B_n - 6^2 C_n}{\cos^6 x} \ldots$$

Hence we have

$$A_{n+1} = -2^2 A_n, \quad B_{n+1} = 2.3A_n - 4^2 B_n,$$

$$C_{n+1} = 4.5B_n - 6^2 C_n \ldots \text{ \&c. \quad \&c.}$$

From these equations we may find by successive substitution the quantities A_n, B_n, &c.

The general expression for $\frac{d^n y}{dx^n}$ is very complicated in this case.

(6) $\quad y = x\epsilon^x$;

$$\therefore \frac{dy}{dx} = (x+1)\epsilon^x,$$

$$\frac{d^2y}{dx^2} = (x+2)\epsilon^x,$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\frac{d^n y}{dx^n} = (x+n)\epsilon^x.$$

(7) $\quad y = x^m \epsilon^x$,

$$\frac{dy}{dx} = (x^m + mx^{m-1})\epsilon^x,$$

$$\frac{d^2y}{dx^2} = \{x^m + 2mx^{m-1} + m.(m-1).x^{m-2}\}\epsilon^x.$$

By Leibnitz' Theorem, see Chap. XIX., we obtain immediately, putting $u = \epsilon^x$, $v = x^m$,

$$\frac{d^n y}{dx^n} = \epsilon^x \left\{x^m + \frac{n}{1} m x^{m-1} + \frac{n.(n-1)}{\Gamma 2} m(m-1).x^{m-2}\ldots\right\}.$$

(8) $y = \epsilon^x \sin x$.

We find similarly by Leibnitz' Theorem,

$$\frac{d^n y}{dx^n} = \epsilon^x \left\{\sin x + \frac{n}{1}\cos x - \frac{n(n-1)}{\Gamma 2}\sin x - \frac{n(n-1)(n-2)}{\Gamma 3}\cos x + \ldots\right\}$$

$$= \epsilon^x \left[\left\{1 - \frac{n.(n-1)}{\Gamma 2} + \frac{n(n-1).(n-2)(n-3)}{\Gamma 4}\ldots\right\}\sin x + \left\{\frac{n}{1} - \frac{n(n-1)(n-2)}{\Gamma 3}\ldots\right\}\cos x\right]$$

$$= \frac{\epsilon^x}{2}\left[\{(1+\sqrt{-1})^n + (1-\sqrt{-1})^n\}\sin x + \frac{1}{\sqrt{-1}}\{(1+\sqrt{-1})^n - (1-\sqrt{-1})^n\}\cos x\right].$$

Now $1 \pm \sqrt{-1} = \sqrt{2}\left(\cos\frac{\pi}{4} \pm \sin\frac{\pi}{4}\sqrt{-1}\right)$;

$$\therefore (1 \pm \sqrt{-1})^n = 2^{\frac{n}{2}}\left(\cos\frac{m\pi}{4} \pm \sin\frac{n\pi}{4}\sqrt{-1}\right);$$

$$\therefore \frac{d^n y}{dx^n} = 2^{\frac{n}{2}}\epsilon^x\left(\sin x\cos\frac{n\pi}{4} + \cos x\sin\frac{n\pi}{4}\right)$$

$$= 2^{\frac{n}{2}}\epsilon^x \sin\left(x + \frac{n\pi}{4}\right).$$

Or thus more generally and simply.

Let $y = \epsilon^{mx}\sin(x + a)$,

then $\dfrac{dy}{dx} = \epsilon^{mx}\{\sin(x+a) + m\cos(x+a)\}$

$$= \frac{\epsilon^{mx}}{\cos\beta}\sin(x+a+\beta),$$

if we put $m = \tan\beta$;

$\therefore$ in a similar manner we find,

$$\frac{d^2y}{dx^2} = \frac{\epsilon^{mx}}{\cos^2\beta}\sin(x+a+2\beta);$$

and $\therefore$ in general $\dfrac{d^ny}{dx^n} = \dfrac{\epsilon^{mx}}{\cos^n\beta}\sin(x+a+n\beta)$;

(9) $y = \tan^{-1}x$,

then $\dfrac{dy}{dx} = \cos^2 y$,

$$\frac{d^2y}{dx^2} = -2\sin y\cos y\frac{dy}{dx} = -\sin 2y\cos^2 y$$

$$= \cos\left(2y + \frac{\pi}{2}\right)\cos^2 y,$$

$$\frac{d^3y}{dx^3} = -2\left\{\cos\left(2y+\frac{\pi}{2}\right)\sin y + \sin\left(2y+\frac{\pi}{2}\right)\cos y\right\}\cos y\frac{dy}{dx}$$

$$= 2\cos(3y+\pi)\cos^3 y$$

....

Now if $u = A\cos(ny+\beta)\cos^n y$,

$$\frac{du}{dx} = -nA\{\cos(ny+\beta)\sin y + \sin(ny+\beta)\cos y\}\cos^{n-1}y\frac{dy}{dx}$$

$$= nA\cos\left\{(n+1)y + \beta + \frac{\pi}{2}\right\}\cos^{n+1}y;$$

from which it is clear, that if we assume

$$\frac{d^ny}{dx^n} = \Gamma(n-1)\cos\left\{ny + (n-1)\cdot\frac{\pi}{2}\right\}\cos^n y,$$

the same law will hold when $n+1$ is put for n, but we have found this law to be true for 2 and 3, therefore it is true in general; and it gives us the n^{th} differential coefficient of $\tan^{-1} x$.

(E.)

(1) Having given $u = \dfrac{ds^3}{d^2y\,dx}$,

where dx is constant, and $ds^2 = dx^2 + dy^2$; to find what u becomes if s be made the independant variable.

We have $$d^2y = dx\,d\left(\frac{dy}{dx}\right);$$

$$\therefore\ d^2y\,dx = d^2y\,dx - dy\,d^2x \ldots (1).$$

Now since $ds^2 = dx^2 + dy^2$, and ds is constant, we have

$$0 = dx\,d^2x + dy\,d^2y \ \ldots (2),$$

$(1)^2 + (2)^2$ gives, observing that $dx^2 + dy^2 = ds^2$,

$$(d^2y\,dx)^2 = (d^2y)^2 + (d^2x)^2.$$

Hence $$u = \frac{ds^3}{\sqrt{(d^2y)^2 + (d^2x)^2}},$$

in which expression s is the independant variable.

(2) Given $u = \dfrac{ds^3}{d^2y\,dx}$,

where dx is constant, and $x = a\cos nt$, $y = b\sin nt$; to make t the independant variable,

$$dx = -na\sin nt\,dt, \quad dy = nb\cos nt\,dt,$$

$$ds^2 = n^2\left(a^2\sin^2 nt + b^2\cos^2 nt\right)dt^2,$$

$$\therefore\ \frac{dy}{dx} = -\frac{b}{a}\cot nt,$$

$$dx\,d^2y = dx^2 d\left(\frac{dy}{dx}\right) = \frac{b}{a}\,\frac{n\,dt}{\sin^2 nt}\,n^2a^2\sin^2 nt\,dt^2$$

$$= n^3ab\,dt^3;$$

$$\therefore\ u = \frac{(a^2\sin^2 nt + b^2\cos^2 nt)^{\frac{3}{2}}}{ab}.$$

(F.)

(1) $y = e^v \tan u + \log(u+v)$,

$$d^2y = \left(e^v\tan u + \frac{1}{u+v}\right)d^2v + \left(\frac{e^v}{\cos^2 u} + \frac{1}{u+v}\right)d^2u$$

$$+\left\{e^v\tan u - \frac{1}{(u+v)^2}\right\}dv^2 + 2\left\{\frac{e^v}{\cos^2 u} - \frac{1}{(u+v)^2}\right\}du\,dv$$

$$+\left\{\frac{2e^v\sin u}{\cos^3 u} - \frac{1}{(u+v)^2}\right\}du^2.$$

(2) $y = u^{u+v} + v^{u+v}$;

put $u + v = z$, then $y = u^z + v^z$;

$$\therefore\ dy = (u^z\log u + v^z\log v)\,dz + zu^{z-1}du + zv^{z-1}dv,$$

and $\therefore$ since $dz = du + dv$,

$$dy = \{u^{u+v}\log u + v^{u+v}\log v + (u+v)\,u^{u+v-1}\}\,du$$

$$+ \{u^{u+v}\log u + v^{u+v}\log v + (u+v)\,v^{u+v-1}\}\,dv.$$

(3) $y\sin x + x\sin y - xy = 0$,

$$y\cos x + \sin y - y + (\sin x + x\cos y - x)\frac{dy}{dx} = 0$$

$$-y\sin x + 2(\cos x + \cos y - 1)\frac{dy}{dx} - x\sin y\left(\frac{dy}{dx}\right)^2$$

$$+ (\sin x + x\cos y - x)\frac{d^2y}{dx^2} = 0,$$

&c. ... &c.,

whence $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$, &c. may be determined.

(G.)

(1) Let $y = \sin^{-1} x$;

then $$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} = 1 + \tfrac{1}{2} x^2 + \frac{1 \cdot 3}{2 \cdot 4} x^4 \ldots\ldots$$

$$\frac{1 \cdot 3 \ldots 2n - 1}{2 \cdot 4 \ldots 2n} x^n + \&c.;$$

$$\therefore \frac{d^{n+1} y}{dx^{n+1}} = \frac{1 \cdot 3 \ldots 2n - 1}{2 \cdot 4 \ldots 2n} . \Gamma n + \text{powers of } x$$

$$= \frac{1 \cdot 3 \ldots 2n - 1}{2 \cdot 4 \ldots 2n} \Gamma n, \text{ when } x = 0.$$

Hence the general term of the expansion of y in powers of x is

$$\frac{1 \cdot 3 \ldots 2n - 1}{2 \cdot 4 \ldots 2n} \cdot \frac{x^{n+1}}{n + 1},$$

and $\therefore$ since $\sin^{-1} x = m\pi$ when $x = 0$, we have

$$\sin^{-1} x = m\pi + \frac{x}{1} + \tfrac{1}{2} \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{x^5}{5} \ldots \&c.$$

(2) We may often obtain developements of this kind more simply in the following manner.

Assume $y = A_0 + A_1 x + A_2 x^2 \ldots \&c.$,

and $\therefore$ $\dfrac{dy}{dx} = A_1 + 2A_2 x + 3A_3 x^2 \ldots \&c.$

Comparing this with the above value of $\dfrac{dy}{dx}$ we find immediately

$$A_1 = 1,\ A_2 = 0,\ A_3 = \tfrac{1}{3}\tfrac{1}{2},\ A_4 = 0,\ A_5 = \tfrac{1}{5}\frac{1 \cdot 3}{2 \cdot 4} \ldots \&c.$$

Also A_0 = the value of y when $x = 0$, which is $m\pi$;

hence $y = m\pi + \dfrac{x}{1} + \tfrac{1}{2}\dfrac{x^3}{3} + \&c. \ldots$ as before.

(3) $$y = \sin x \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Assume $$y = A_0 + A_1 x + A_2 x^2 \ldots\ldots\ldots\ldots (2),$$

then from (1) we have

$$\frac{d^2 y}{dx^2} = -\sin x = -A_0 - A_1 x - A_2 x^2 \ldots\ldots (3),$$

and from (2) we have

$$\frac{d^2 y}{dx^2} = 1 \,.\, 2\, A_2 + 2 \,.\, 3\, A_3 x + 3 \,.\, 4\, A_4 x^2 \ldots\ldots (4).$$

Now $A_0 = y$ when $x = 0$; $\therefore A_0 = 0$;

and $A_1 = \dfrac{dy}{dx} = \cos x$ when $x = 0$; $\therefore A_1 = 1$;

and therefore, comparing (3) and (4), we have

$$A_2 = 0,\ A_3 = -\frac{1}{\Gamma 3},\ A_4 = 0,\ A_5 = -\frac{A_3}{4 \,.\, 5} = \frac{1}{\Gamma 5} \ldots \&c.;$$

$$\therefore \sin x = \frac{x}{\Gamma 1} - \frac{x^3}{\Gamma 3} + \frac{x^5}{\Gamma 5} \ldots \&c.$$

(4) If we differentiate the result just obtained, we find

$$\cos x = 1 - \frac{x^2}{\Gamma 2} + \frac{x^4}{\Gamma 4} \ldots \&c.$$

(5) If $\tan y = a \tan x$, to expand y in powers of x.

(6) If $\sin y = a \sin (y + x)$, to expand y in powers of x.

(H.)

(1) To expand $\cot x$ in powers of x.

$\cot x = \infty$ when $x = 0$; therefore there are negative powers in the developement, and Taylor's series fails. We may proceed as follows:

$$\cot x = \frac{\cos x}{\sin x} = \frac{1 - \dfrac{x^2}{\Gamma 2} + \dfrac{x^4}{\Gamma 4} \ldots\ldots}{\dfrac{x}{\Gamma 1} - \dfrac{x^3}{\Gamma 3} \ldots \text{ \&c.}}$$

$$= \frac{1}{x} \cdot \frac{1 - \dfrac{x^2}{\Gamma 2} \ldots\ldots \text{ \&c.}}{1 - \dfrac{x^2}{\Gamma 3} \ldots\ldots \text{ \&c.}};$$

assume this to be equal to

$$\frac{1}{x}(A + Bx + Cx^2 \ldots);$$

then we have

$$(A + Bx + Cx^2 \ldots)(1 - \frac{x^2}{\Gamma 3} \ldots) = 1 - \frac{x^2}{\Gamma 2} \ldots$$

$$\text{or } A + Bx + \left(C - \frac{A}{\Gamma 3}\right) x^2 \ldots = 1 - \frac{x^2}{\Gamma 2} \ldots$$

$$\therefore A = 1, \quad B = 0, \quad C = \frac{1}{\Gamma 3} - \frac{1}{\Gamma 2} = -\tfrac{1}{3}, \text{ \&c.}$$

$$\therefore \cot x = \frac{1}{x}(1 - \frac{x^2}{3} \ldots \text{ \&c.}).$$

(2) Let $y = (e^x - 1)^{\frac{1}{3}}$.

Here $\dfrac{dy}{dx} = \tfrac{1}{3} e^x (e^x - 1)^{-\frac{2}{3}} = \infty$ when $x = 0$:

therefore fractional powers occur in the developement, and Taylor's series fails. We may proceed as follows:

$$e^x - 1 = \frac{x}{\Gamma 1} + \frac{x^2}{\Gamma 2} \ldots$$

$$\therefore y = x^{\frac{1}{3}}(1 + \frac{x}{\Gamma 2} + \frac{x^2}{\Gamma 3} \ldots)^{\frac{1}{3}}$$

$$= x^{\frac{1}{3}}(1 + \tfrac{1}{3}\frac{x}{\Gamma 2} \ldots \text{ \&c.}),$$

by the binomial theorem.

(3) $$y = 1 + x\log(e+\sqrt{x}).$$

Here $\dfrac{d^2y}{dx^2} = \infty$, and $y = 1 + x + \dfrac{x^{\frac{3}{2}}}{e} \dots$ &c.

(I.)

The following remark is of some importance, as it will often enable us to simplify the process explained in (145).

Suppose that we wish to expand y in powers of x, having given a relation between y and x; then if $y = a$ when $x = 0$, we assume $y = a + ux^m$, as is explained in (145).

Now suppose that the result of this substitution comes out in the form

$$U_1x^{r_1} + U_2x^{r_2} + U_3x^{r_3} + \text{\&c.} = 0,$$

and suppose that we can see by inspection that one of the indices r_2 must be greater than another r_1 whatever value m may have, (m of course is always positive); then we may *immediately reject* the term $U_2x^{r_2}$, since it cannot be one of the terms containing the *lowest* power of x, and therefore cannot affect the result, as is evident from the example in (145).

(1) Let $y^4 - x^2y^2 + ax^2y - a^2x^2 = 0$;

here $y = 0$ when $x = 0$;

assume $\therefore\ y = ux^m$; then

$$u^4x^{4m} - u^2x^{2m+2} + aux^{m+2} - a^2x^2 = 0.$$

Now here $2m + 2$ must be $> m + 2$ whatever m be, and $m + 2$ must be > 2; hence we may immediately reject the terms u^2x^{2m+2} and aux^{m+2}; and $\therefore$ we have simply

$$u^4x^{4m} - a^2x^2 = 0,$$

which gives $4m = 2$, and $\therefore\ m = \frac{1}{2}$, and $\therefore\ u = \pm\sqrt{a}$,

$$\text{and } \therefore\ y = \pm\sqrt{a}\,.\,x^{\frac{1}{2}} + R.$$

Of course we must not reject these terms if we proceed to find the second and higher terms of the expansion.

(2) $$y^4 - x^2y^2 + axy^2 - a^2x^3 = 0;$$

$$\therefore\ u^4x^{4m} - u^2x^{2m+2} + au^2x^{2m+1} - a^2x^3;$$

or since $2m+2$ is $>2m+1$,

$$u^4x^{4m} - au^2x^{2m+1} - a^2x^3.$$

Suppose that m is >1, then $4m$ is $>2m+1$, and $2m+1$ is >3, which cannot be; therefore m is <1; $\therefore$ 3 is $>2m+1$; $\therefore$ a^2x^3 is to be rejected, and $\therefore$ we have

$$4m = 2m+1, \text{ and } \therefore\ m = \tfrac{1}{2}, \text{ and } \therefore\ u = \pm\sqrt{a};$$

$$\therefore\ y = \pm\sqrt{a}\,.\,x^{\frac{1}{2}} + R.$$

(J.)

(1) Let $y = x\log x$; to find the limiting value of y when x approaches zero.

We have $y = \dfrac{x}{\dfrac{1}{\log x}} = \dfrac{0}{0}$ when $x = 0$;

$\therefore$ by Lemma XX., y and $\dfrac{dx}{d\dfrac{1}{\log x}}$

have the same limiting value when x approaches 0.

Now $$\frac{dx}{d\dfrac{1}{\log x}} = -x(\log x)^2 = -\frac{y^2}{x};$$

$\therefore$ y and $-\dfrac{y^2}{x}$ have the same limiting value when x approaches 0;

therefore, multiplying each of these quantities by $\frac{x}{y}$, x and $-y$ have the same limiting value when x approaches zero, and therefore 0 is the limiting value of y.

Here we assume, that if any functions U and V have the same limiting value when x approaches a, the same is true of WU and WV, W being any other function; which is evidently true, since if A be the limiting value of U and V, and B that of W, then by Lemma VIII, BA and BA are the limiting values of WU and WV.

(2) Let $y = xe^{-x}$; to find the limiting value of y when x approaches ∞.

$$y = \frac{e^{-x}}{\frac{1}{x}} = \frac{0}{0} \text{ when } x = \infty;$$

$\therefore$ y and $\frac{de^{-x}}{d\frac{1}{x}}$ have the same limiting value when x approaches ∞.

Now $\frac{de^{-x}}{d\frac{1}{x}} = x^2 e^{-x} = \frac{y^2}{e^{-x}}$;

$\therefore$ y and e^{-x} have the same limiting value when x approaches ∞, therefore the limiting value of y is 0.

(3) Let $y = (\sin x)^{\tan x}$; to find the limiting value of y when x approaches 0.

$$\log y = \tan x \log \sin x = \frac{1}{\cos x} z \log z,$$

if we put $\sin x = z$; hence, since z approaches zero when x does, and since the limiting value of $\cos x$ is 1, and that of $z \log z$ zero, the limiting value of $\log y$ is zero, and therefore that of y is unity.

(K.)

(1) To inscribe the greatest rectangle in a semi-circle.

Let $CM = x$, $MP = y$ (fig. 33), then $2xy$ is the area of the inscribed rectangle; hence we have

$$d(2xy) = 0; \text{ and } \therefore d(x^2y^2) = 0;$$

$$\text{or } d\{x^2(a^2 - x^2)\} = 0;$$

$$\therefore 2a^2x - 4x^3 = 0,$$

which is satisfied by $x = \frac{a}{\sqrt{2}}$ and $x = 0$. $x = \frac{a}{\sqrt{2}}$ evidently gives a maximum.

(2) To inscribe the greatest semi-ellipse in an isosceles triangle. (fig 34.)

Let $CM = x$, $MP = y$, then if $CE = h$, $CF = k$, we have

$$h = \frac{a^2}{x}, \quad k = \frac{b^2}{y};$$

$$\therefore \frac{a^2}{h^2} + \frac{b^2}{k^2} = 1 \ldots\ldots (1).$$

Also πab is to be a maximum;

$$\text{and } \therefore d(ab) = 0;$$

$$\therefore bda + adb;$$

$$\text{and by (1),} \qquad \frac{ada}{h^2} + \frac{bdb}{k^2} = 0;$$

$$\therefore \frac{a}{bk^2} = \frac{b}{ak^2},$$

which, along with (1), gives a and b.

(3) To find the greatest triangle that can be inscribed in a given circle.

ABC (fig. 35) is a maximum, supposing AB invariable, if $AC = BC$, as may be easily shewn. Therefore *any* two

sides of the maximum triangle must be equal; therefore it must be an equilateral triangle.

(4) To find the greatest quadrilateral figure that can be circumscribed about a circle.

We may prove similarly that it must be a square.

(5) To find the greatest triangle which has a given perimeter.

(6) To find the greatest polygon that has a given perimeter.

Suppose all the sides but three invariable.

(L.)

(1) The normal to any curve is in general the greatest or least line which can be drawn to a given point to it.

Let xy be any point on the curve, $x'y'$ any other point, and R the distance between them; then

$$R^2 = (x - x')^2 + (y - y')^2,$$

$$R\,dR = (x - x')\,dx + (y - y')\,dy,$$

but $dR = 0$ if R be a maximum or minimum; and therefore

$$x' - x + (y' - y)\frac{dy}{dx} = 0,$$

which shews that $x'y'$ is a point of the normal at xy.

(2) To draw a tangent to a given curve, cutting off the greatest or least area from the space included between the co-ordinate axes.

The equation to the tangent being

$$(y' - y)\,dx - (x' - x)\,dy = 0,$$

the portions it cuts off the axes are

$$\beta = \frac{y\,dx - x\,dy}{dx}, \quad \text{and } \alpha = -\frac{x\,dy - y\,dx}{dy};$$

$$\text{and } \frac{\alpha\beta}{2} \text{ or } -\frac{(ydx - xdy)^2}{2dxdy}$$

is to be a maximum.

Therefore, considering dx as constant, we have

$$d\left\{\frac{(ydx - xdy)^2}{dy}\right\} = 0,$$

$$\text{or } \frac{-2(ydx - xdy)\,xd^2ydy - (ydx - xdy)^2 d^2y}{dy^2} = 0,$$

$$\text{which gives } 2x = -\frac{ydx - xdy}{dy} = -a;$$

$$\therefore\ x = \frac{a}{2}:$$

this equation shews that the portion of the tangent intercepted between the co-ordinate axes is bisected at the point of contact when it cuts off a maximum or minimum area.

(3) In the ellipse, if we assume $x = a\cos t$, then we have $y = b\sin t$, in virtue of the equation

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}.$$

Hence we may always suppose in the ellipse, that

$$x = a\cos t, \text{ and } y = b\sin t.$$

If we describe a circle on the major-axis as diameter, and if we produce the ordinate MP to meet this circle in Q, then it is evident that $\angle QCA = t$, for

$$\cos QCA = \frac{CM}{MQ} = \frac{x}{a} = \cos t.$$

This transformation is often useful.

Differentiating, we have

$$dx = -a\sin t\,dt, \quad dy = b\cos t\,dt;$$

$$\therefore\ ds = \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\,.\,dt,$$

$$\tan\psi = -\frac{b}{a}\cot t,$$

$$\cos\psi = \frac{1}{\sqrt{1 + \dfrac{b^2}{a^2}\cot^2 t}}.$$

Equation to tangent is

$$\frac{x\cos t}{a} + \frac{y\sin t}{b} = 1.$$

(4) To find the locus of the intersection of the tangent and a perpendicular upon it from the focus of an ellipse.

The equation to the tangent is

$$y = -\frac{b}{a}\cot t\,.\,x + \frac{b}{\sin t}.$$

The equation to the perpendicular upon this from the focus is

$$y = \frac{a}{b}\tan t\,.\,(x - ae),$$

which equations may be put in the form

$$ay\sin t + bx\cos t = ab,$$

$$by\cos t - ax\sin t = -a^2 e\sin t;$$

therefore, squaring and adding these equations,

$$(x^2 + y^2)(a^2\sin^2 t + b^2\cos^2 t) = a^2(b^2 + a^2e^2\sin^2 t)$$

$$= a^2(b^2\cos^2 t + a^2\sin^2 t).$$

Since $a^2e^2 = a^2 - b^2$;

$$\therefore\ x^2 + y^2 = a^2;$$

hence the locus required is the circle described on the major axis as diameter.

(5) To find what relation must hold between α and β in the equation

$$\frac{x_1}{\alpha} + \frac{y_1}{\beta} = 1 \ldots (1).$$

So that it shall be the equation to a tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ldots (2).$$

It is easy to shew that the equation to the tangent to (2) is

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1.$$

Comparing this with (1) we have

$$\alpha = \frac{a^2}{x}, \quad \beta = \frac{b^2}{y};$$

$$\therefore \frac{x}{a} = \frac{a}{\alpha}, \quad \frac{y}{b} = \frac{b}{\beta};$$

$$\therefore \text{by } (2) \quad \frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} = 1,$$

which is the relation required.

(6) To find the curve in which the subnormal is a constant c.

We have subnormal $= y\dfrac{dy}{dx} = c$;

$$\therefore 2ydy - 2cdx = 0;$$

$$\therefore d(y^2 - 2cx) = 0.$$

Now in general if $du = 0$, u must be a constant;

$\therefore y^2 = 2cx$ + some constant, which is the equation to a parabola.

(7) To find the curve in which the subtangent is twice the abscissa.

We have $y\dfrac{dx}{dy} = 2x$;

$$\therefore \frac{dx}{x} - 2\frac{dy}{y} = 0;$$

$$\therefore d(\log x - 2\log y) = 0;$$

$$\therefore \log x - 2\log y = \text{constant} = \log C \text{ suppose};$$

$$\therefore \frac{x}{y^2} = C, \text{ which is the equation to a parabola.}$$

(8) In the ellipse, if n be the normal and p the perpendicular from the center on the tangent,

$$pn = b^2.$$

For $p = \dfrac{ydx - xdy}{ds}$, $\quad n = y\dfrac{ds}{dx}$;

$$\therefore pn = y^2 - xy\frac{dy}{dx}$$

$$= y^2 + \frac{b^2x^2}{a^2}, \text{ since } \frac{dy}{dx} = -\frac{b^2x}{a^2y}$$

$$= b^2.$$

(M.)

(1) Let the equation to the curve be

$$a^3y = x^4 - 6ax^3 + 12a^2x^2,$$

then $a^3\dfrac{d^2y}{dx^2} = 12(x^2 - 3ax + 2a^2) = 12(x - a)(x - 2a)$;

$\because$ the sign of $\dfrac{d^2y}{dx^2}$ is $+$ when x is $< a$,

$-$ when x is $> a$ and $< 2a$,

and $+$ when x is $> 2a$.

Hence the concavity of the curve is turned upwards when x is $< a$, downwards when x is between a and $2a$, and upwards when x is $> 2a$. And there are two points of contrary flexure, one when $x = a$, and another when $x = 2a$.

(2) Let the equation to the curve be

$$y = \frac{a^2 x}{(x - a)^2},$$

$$\text{then} \quad \frac{d^2 y}{dx^2} = \frac{2a^2 (x + 2a)}{(x - a)^4}.$$

Here there is a change of flexure when x passes through the value $- 2a$, but none when x passes through the value a.

(N.)

(1) To find ρ in the ellipse.

We have $x = a \cos t$, $\quad y = b \sin t$, $\{$page 217 (3),$\}$

$$\rho = \frac{ds^3}{dx d^2 y - dy d^2 x}$$

$$= \frac{(a^2 \sin^2 t + b^2 \cos^2 t)^{\frac{3}{2}}}{ab \sin^2 t + ab \cos^2 t}$$

$$= \frac{1}{ab} \left(\frac{a^2}{b^2} y^2 + \frac{b^2}{a^2} x^2 \right)^{\frac{3}{2}}.$$

(2) If n be the normal,

$$n = y \frac{ds}{dx};$$

$$\therefore \; ds = \frac{n dx}{y} = - \frac{a}{b} n dt;$$

$$\therefore \; \rho = - \frac{1}{ab} \cdot \frac{a^3}{b^3} n^3 = - \frac{n^3}{\left(\frac{b^2}{a}\right)^2}.$$

The negative sign indicates that the concavity of the curve is turned downwards, in which case $\frac{d\psi}{ds}$ is negative.

(3) To find ρ in the parabola.

Let us use the equation

$$y = \frac{x^2}{4m},$$

$$\text{then} \quad dy = \frac{x}{2m}\, dx;$$

$$\therefore\ ds^2 = \left(1 + \frac{x^2}{4m^2}\right) dx^2 = \left(1 + \frac{y}{m}\right) dx^2,$$

$$\text{and} \quad d^2y = \frac{dx^2}{2m} \quad (dx \text{ being constant});$$

$$\therefore\ \rho = \frac{ds^3}{dx\,d^2y} = 2m\left(1 + \frac{y}{m}\right)^{\frac{3}{2}}.$$

(O.)

(1) To find the evolute of the ellipse.

We have $\qquad \alpha = x - \rho \sin\psi,$

$$\beta = y + \rho\cos\psi,$$

$$\text{or} \quad \alpha = x - \frac{ds^2 dy}{dx\,d^2y - dy\,d^2x},$$

$$\beta = y + \frac{ds^2 dx}{dx\,d^2y - dy\,d^2x}.$$

Hence in the ellipse,

$$\alpha = a\cos t - \frac{(a^2\sin^2 t + b^2\cos^2 t)\, b\cos t}{ab}$$

$$= \frac{1}{a}(a^2 - a^2\sin^2 t - b^2\cos^2 t)\cos t$$

$$= \frac{a^2 - b^2}{a}\cos^3 t.$$

Similarly $\beta = \dfrac{b^2 - a^2}{b} \sin^3 t.$

Hence, since $\sin^2 t + \cos^2 t = 1$, we have

$$\left(\frac{\alpha}{a_1}\right)^{\frac{2}{3}} + \left(\frac{\beta}{b^1}\right)^{\frac{2}{3}} = 1,$$

$$\text{when } a_1 = \frac{a}{a^2 - b^2}, \text{ and } b_1 = \frac{b}{b^2 - a^2};$$

which is the equation to the evolute of an ellipse. It is represented by $UVWY$, (fig. 36).

(2) To find the evolute to the parabola.

We have
$$y = \frac{x^2}{4m},$$

and supposing dx constant,

$$\alpha = x - \frac{ds^2 dy}{d^2 y\, dx},$$

$$\beta = y + \frac{ds^2}{d^2 y};$$

$$\text{also } ds^2 = \left(1 + \frac{y}{m}\right) dx^2,$$

$$dy = \frac{x}{2m} dx, \quad d^2 y = \frac{dx^2}{2m};$$

$$\therefore \alpha = x - 2m\left(1 + \frac{y}{m}\right)\frac{x}{2m} = -\frac{xy}{m} = -\frac{x^3}{4m^2},$$

$$\beta = y + 2m\left(1 + \frac{y}{m}\right) = 3y + 2m;$$

$$\therefore x = -(4m^2 \alpha)^{\frac{1}{3}}, \quad y = \frac{\beta - 2m}{3};$$

$$\therefore \frac{\beta - 2m}{3} = \frac{(4m^2 \alpha)^{\frac{2}{3}}}{4m},$$

$$\text{or } \beta = 2m + \frac{m^{\frac{1}{3}}}{2^{\frac{1}{3}}}\alpha^{\frac{2}{3}},$$

which is the equation to the evolute of a parabola

It is represented by UWV (fig. 37).

(P.)

(1) Let the equation to the curve be

$$r = a e^{m\theta},$$

$$\text{then} \quad dr = m a e^{m\theta} d\theta,$$

$$\tan\psi = \frac{r d\theta}{dr} = m.$$

Hence in this curve, which is called the logarithmic spiral, the tangent always makes the same angle with the radius vector.

$$ds^2 = r^2 d\theta^2 + dr^2 = a^2 e^{2m\theta}(1 + m^2)\, d\theta^2;$$

$$\therefore\ ds - a\sqrt{1 + m^2}\,.\,e^{m\theta} d\theta = 0,$$

$$d\left(s - a\frac{\sqrt{1 + m^2}}{m}e^{m\theta}\right) = 0;$$

$$\therefore\ s - a\frac{\sqrt{1 + m^2}}{m}e^{m\theta} = \text{some constant, } C \text{ suppose:}$$

if $s = 0$ when $\theta = 0$, then

$$0 - a\frac{\sqrt{1 + m^2}}{m} = C;$$

$$\therefore\ s = \frac{\sqrt{1 + m^2}}{m}(r - a).$$

(2) The polar equation to an ellipse may be put in the form

$$\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} = \frac{1}{r^2} \ldots\ldots\ldots\ldots\ldots\ldots(1);$$

$$\therefore \left(\frac{1}{a^2} - \frac{1}{b^2}\right) \cos\theta \sin\theta = \frac{1}{r^2} \cdot \frac{dr}{r\,d\theta} = \frac{1}{r^2} \cot\phi \ldots (2).$$

Now (1) gives us

$$\left(\frac{1}{a^2} - \frac{1}{b^2}\right) \cos^2\theta = \frac{1}{r^2} - \frac{1}{a^2},$$

$$\left(\frac{1}{a^2} - \frac{1}{b^2}\right) \sin^2\theta = \frac{1}{b^2} - \frac{1}{r^2};$$

therefore squaring each side of (2) and substituting these values, we have

$$\left(\frac{1}{r^2} - \frac{1}{a^2}\right)\left(\frac{1}{b^2} - \frac{1}{r^2}\right) = \frac{1}{r^4} \cot^2\phi,$$

$$\text{or} \quad r^4 - (a^2 + b^2) r^2 + \frac{a^2 b^2}{\sin^2\phi} = 0,$$

which equation gives us r, the length of the diameter which makes an angle ϕ with the tangent at its extremity.

Hence, since conjugate diameters are parallel to tangents at their extremities, the two values of r got from this equation are the lengths ($a'b'$ suppose) of the two conjugate diameters which make an angle ϕ with each other. We have therefore

$$a'^2 + b'^2 = a^2 + b^2,$$

$$a'^2 . b'^2 = \frac{a^2 b^2}{\sin^2\phi}, \text{ or } a'b' \sin\phi = ab,$$

two well known properties of the ellipse.

(3) In the spiral $r = ae^{m\theta}$,

$$p = \frac{r^2 d\theta}{ds} = \frac{r^2 d\theta}{\sqrt{1 + m^2} . r\,d\theta};$$

$$\therefore p = \frac{r}{\sqrt{1 + m^2}} \ldots (1);$$

$$\therefore dp = \frac{dr}{\sqrt{1 + m^2}};$$

$$\therefore\ \rho = \frac{r\,dr}{dp} = \sqrt{1+m^2}\,.\,r.$$

If O be the extremity of ρ, and $SO = r_1$, then

$$\begin{aligned} r_1^2 &= \rho^2 + r^2 - 2r\rho\cos SPO \\ &= \rho^2 + r^2 - 2p\rho \\ &= m^2r^2; \end{aligned}$$

$$\therefore\ r_1 = mr.$$

Also if p_1 be the perpendicular from S on PO, we have

$$p_1^2 + p^2 = r^2; \quad \therefore\ p_1 = \frac{m}{\sqrt{1+m^2}}\,r;$$

$$\therefore\ p_1 = \frac{r_1}{\sqrt{1+m^2}}.$$

Now r_1 and p_1 are evidently the radius vector of the evolute and perpendicular upon its tangent; hence, comparing this relation between p_1 and r_1 with (1), it follows that the evolute to a logarithmic spiral is a similar spiral.

In general we may find the equation between p_1 and r_1 by eliminating p, ρ, and r, between the equations

$$\left.\begin{aligned} p &= f(r) \\ \rho &= \frac{r\,dr}{dp} = \frac{r}{f'(r)} \\ r_1^2 &= r^2 + \rho^2 - 2\rho p \\ p_1^2 + p^2 &= r^2 \end{aligned}\right\},$$

$p = f(r)$ being the equation to the given curve between p and r. In this manner we may obtain the equation to the evolute of a spiral.

(S.)

(1) Let the given equation be

$$a^2y^2 = \frac{x^2(x-a)^2(x+2b)}{x+b};$$

$$\therefore\ ay = \pm x(x-a)\sqrt{\frac{x+2b}{x+b}};$$

then we have the following table.

x	y	$\frac{dy}{dx}$
	$\pm$	
$-2b$	0	∞
	imposs.	
$-b$	∞	∞
	$\pm$	
0	0	$\pm\sqrt{2}$
	$\pm$	
a	0	$\pm\sqrt{\frac{a+2b}{a+b}}$
	$\pm$	
∞	∞	∞

Hence the curve is represented by fig. 38.,

$$AB = a, \quad AC = b, \quad AD = 2b.$$

(2) Let the given equation be

$$y = \frac{a(x-a)^2}{x^2}.$$

See fig. 39., $AB = a$, $AC = a$.

(U.)

(1) To determine the position and nature of the multiple points of the curve,

$$U = y^4 + x^4 - 2a^2y^2 - 2a^2x^2 + a^4 = 0 \;\ldots\; (1),$$

$$\left.\begin{aligned} d_x U &= 4x^3 - 4a^2x = 0 \\ d_y U &= 4y^3 - 4a^2y = 0 \end{aligned}\right\}\ldots\ldots\ldots\ldots (2),$$

which equations are satisfied by any two of the values

$$x = 0 \text{ or } \pm a, \quad \text{and } y = 0 \text{ or } \pm a.$$

Now if $x = 0$, $y = \pm a$ in virtue of (1),

and if $x = \pm a$, $y = 0$ or $\pm\sqrt{2}\,.\,a$.

Hence the only values of x and y which satisfy (1) and (2) are

$$\left.\begin{aligned} x &= 0 \\ y &= a \end{aligned}\right\}(\alpha), \quad \left.\begin{aligned} x &= 0 \\ y &= -a \end{aligned}\right\}(\beta), \quad \left.\begin{aligned} x &= a \\ y &= 0 \end{aligned}\right\}(\gamma), \quad \left.\begin{aligned} x &= -a \\ y &= 0 \end{aligned}\right\}(\delta).$$

Now
$$d_x^2 U = 12x^2 - 4a^2,$$
$$d_x d_y U = 0,$$
$$d_y^2 U = 12y^2 - 4a^2.$$

Hence, using the notation in (161), we have

$A = 0$, $B = 0$,

$C = -4a^2$ if $x = 0$, or $8a^2$ if $x = \pm a$,

$D = 0$,

$E = -4a^2$ if $y = 0$, or $8a^2$ if $x = \pm a$.

Hence for the values (α) and (β), we have

$$-4a^2 + 8a^2u_0^2 = 0, \text{ and } \therefore\ u_0 \text{ or } \frac{dy}{dx} = \pm\frac{1}{\sqrt{2}},$$

and for the values (γ) and (δ), we have

$$8a^2 - 4a^2u_0 = 0, \quad \text{and} \therefore u_0 \text{ or } \frac{dy}{dx} = \pm\sqrt{2}.$$

Hence if we take $AB = AE = AC = AD = a$ (fig. 40), there is a double point at B, at C, at D, and at E, as is represented in the figure.

This is an example which may be very easily solved by the method in (156).

For in this case

$$p = -\frac{x^3 - a^2x}{y^3 - a^2y};$$

$$\therefore p = -\frac{3x^2 - a^2}{(3y^2 - a^2)p};$$

$$\therefore \text{ if } x = 0, \quad \text{and } y = \pm a,$$

$$p^2 = \tfrac{1}{2},$$

$$\text{and if } x = \pm a, \quad \text{and } y = 0,$$

$$p^2 = 2.$$

In general the method given in (159) ought to be used, only when we wish to find $\frac{dy}{dx}$ for the values $x = 0$, $y = 0$, or when more than two differentiations are necessary to find $\frac{dy}{dx}$, in which cases it is simpler than the common method.

(2) Let $U = ay^3 - bx^2y + x^4 = 0 \ldots (1)$,

then $d_xU = -2bxy + 4x^3 = 0 \ \ldots (2)$,

$d_yU = 3ay^2 - bx^2 = 0 \ \ldots\ldots\ldots (3)$.

(2) gives $x = 0$ or $y = \frac{2x^2}{b}$,

if $x = 0$ $y = 0$ in virtue of (3) and these values satisfy (1), if $y = \frac{2x^2}{b}$ (3) becomes

$$\frac{12a}{b^2}x^4 - bx^2 = 0,$$

$$\text{which gives } x^2 = \frac{b^3}{12a} \text{ and } \therefore y = \frac{b^2}{6a}:$$

these values do not satisfy (1), and are therefore to be rejected. We have therefore to consider only the values $x = 0$, $y = 0$.

Hence putting $y = ux$ in (1), we have

$$au^3 - bu + x = 0,$$

$$\text{and } \therefore au_0^3 - bu_0 = 0;$$

$$\therefore \frac{dy}{dx} = 0 \text{ or } \pm\sqrt{\frac{b}{a}},$$

which indicates a triple point at the origin.

(3) To examine the nature of the curve

$$y^5 + ax^4 - b^2xy^2 = 0$$

in the immediate vicinity of the origin.

Assume $y = ux^m$, then

$$ux^{5m} + a^5x^4 - b^2u^2x^{2m+1} = 0.$$

Suppose $5m = 4$; $\therefore m = \frac{4}{5}$, and $\therefore 2m + 1 < 4$; which will not answer.

Suppose $5m = 2m + 1$; $\therefore m = \frac{1}{3}$, and $\therefore 4 > 2m + 1$; which will answer, and gives

$$u^5 - b^2u^2 = 0, \text{ and } \therefore u = b^{\frac{2}{3}}.$$

Suppose $4 = 2m + 1$; $\therefore m = \frac{3}{2}$, and $\therefore 5m > 4$; which will answer, and gives

$$a - b^2u^2 = 0, \text{ and } \therefore u = \pm\frac{\sqrt{a}}{b}.$$

Hence we have

$$y = b^{\frac{2}{3}}x^{\frac{1}{3}} + R,$$

$$\text{and } y = \pm \frac{\sqrt{a}}{b} x^{\frac{3}{2}} + R'.$$

The former of these represents the portion PAP' of the curve (fig. 41), and the latter the portion QAQ'.

We rejected the value $5m = 4$, because it would give $u = \infty$ when $x = 0$; this value belongs to the infinite branches of the curve; for putting $5m = 4$, and $\therefore m = \frac{4}{5}$, and assuming $x = \frac{1}{z}$, we have

$$(u^5 + a)\frac{1}{z^4} - b^2 u^2 \frac{1}{z^{\frac{13}{5}}} = 0,$$

$$\text{or } u^5 + a - b^2 u^2 z^{\frac{7}{5}} = 0,$$

which gives the limiting value of $u = -a^{\frac{1}{5}}$ when z approaches 0;

$$\therefore y = -a^{\frac{1}{5}}\frac{1}{z^{\frac{4}{5}}} + R$$

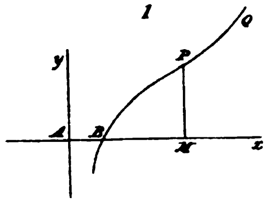
$$= -a^{\frac{1}{5}}x^{\frac{4}{5}} + R,$$

where $R = 0$, when $x = \infty$;

$\therefore y = -a^{\frac{1}{5}}x^{\frac{4}{5}}$ very nearly for large values of x;

$\therefore \frac{dy}{dx} = 0$ when $x = \infty$, as is represented in the figure.

1



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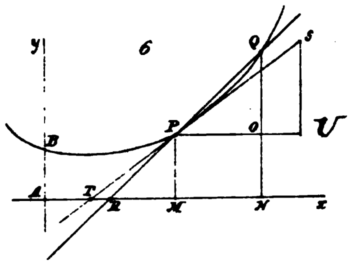

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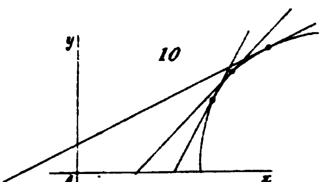

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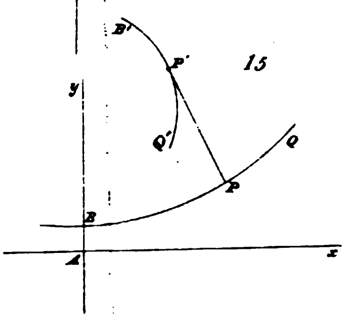

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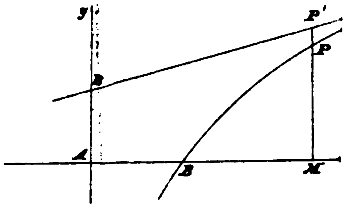
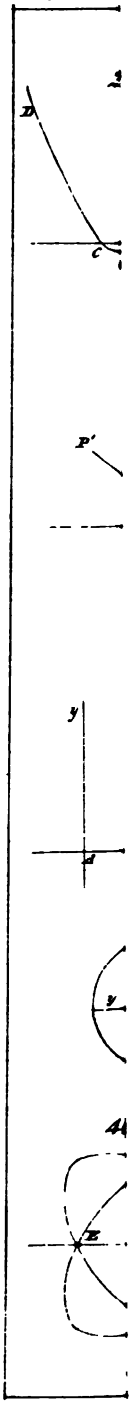
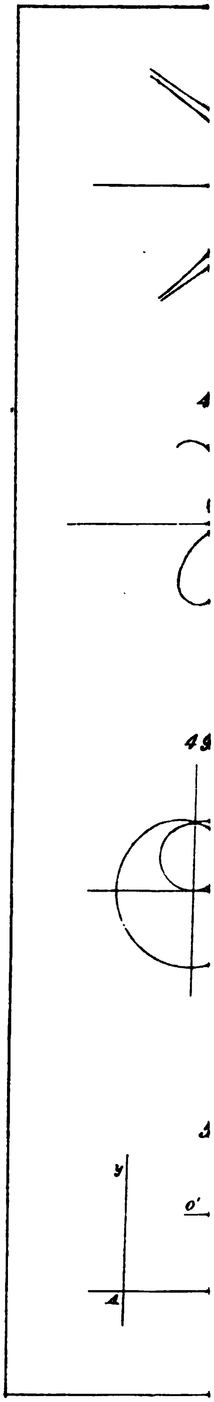
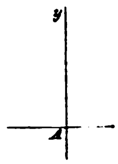


O'Brien

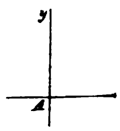


O'Brien's Calc

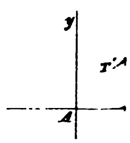


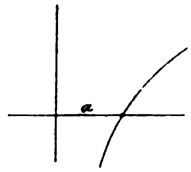


6A

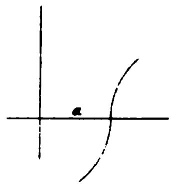


6b

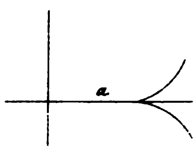




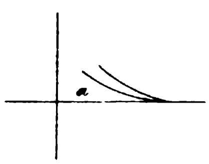
x	y	$\frac{dy}{dx}$
a	$-$ 0 $+$	$+m$



a	$-$ 0 $+$	∞



a	*imposs:* 0 $\pm$	0



a	$+\ +$ 0 *imposs:*	0

O'Brien's Calculus.